Mixed motivic sheaves (and weights for them) exist if 'ordinary' mixed motives do

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Abstract

The goal of this paper is to prove: if certain 'standard' conjectures on motives over algebraically closed fields hold, then over any 'reasonable' S there exists a *motivic* t-structure for the category $DM_c(S)$ of relative Voevodsky's motives (being more precise, for the Beilnson motives described by Cisinski and Deglise). If S is an equicharacteristic scheme, then the heart of this t-structure (the category of *mixed motivic sheaves* over S) is endowed with a *weight filtration* with semi-simple factors. We also prove a certain 'motivic decomposition theorem' (assuming the conjectures mentioned) and characterize semi-simple motivic sheaves over S in terms of those over its residue fields.

Our main tool is the theory of *weight structures*. We actually prove somewhat more than the existence of a weight filtration for mixed motivic sheaves: we prove that the motivic t-structure is *transversal* to the Chow weight structure for $DM_c(S)$ (that was introduced previously by D. Hébert and the author). We also deduce several properties of mixed motivic sheaves from this fact. Our reasoning relies on the degeneration of *Chow-weight spectral sequences* for 'perverse étale homology' (that we prove unconditionally); this statement also yields the existence of the *Chow-weight filtration* for such (co)homology that is strictly restricted by ('motivic') morphisms.

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Introduction

Famous conjectures of Beilinson (see §5.10A in [Bei87]) predict the existence of an abelian category $MM(S)$ of *mixed motivic sheaves* over a (more or less, arbitrary) scheme S. This category should be endowed with a so-called *weight filtration* whose factors are semi-simple; it should possess an exact realization to the category of perverse ($\mathbb{Q}_l$-) étale sheaves. The goal of this paper is to deduce these conjectures from certain 'standard' conjectures on motives over algebraically closed fields.

Now we explain this in more detail. It is widely believed that $MM(S)$ should be the heart of a certain (motivic) t-structure for some triangulated category of (Voevodsky's) motives over S. In this paper we consider this question for the category $DM_c(S)$ of (constructible) Beilinson motives over (an equicharacteristic scheme) S as described in [CiD09], and prove that a ('nice') motivic t-structure exists for it if it exists for Voevodsky's motives

over algebraically closed fields. Recall here: already the latter assumption requires certain very hard 'standard' conjectures (especially for positive characteristic fields; see §4.1 below and §2 of [Han99] for a discussion of those), yet it is nice to know that passing to relative motives in this matter conceals no additional difficulties. Note also: the paper [CoH00] relies on the same conjectures that we need for our results, whereas in ibid. only the properties of (certain) 'pure' relative motivic sheaves (and only for S being a variety over a characteristic 0 field) are established. In particular, we prove a certain motivic version of the Decomposition Theorem for perverse sheaves (see §4.2) that is stronger than the corresponding result of [CoH00]. We also characterize simple mixed motivic sheaves (those are certainly 'pure') in terms of those over the residue fields of S. Certainly, the results of [CiD09] are crucial for our success here.

Now we describe our central results in more detail, and also mention the main prerequisites for their proofs.

Our first central result is the following one. Suppose that for some fixed prime l and for any universal domain K (of characteristic distinct from l) there exists a t-structure t_{MM} for the category $DM_{gm}(K)$ of Voevodsky's motives over K that is strictly compatible with the (canonical) t-structure for $\mathbb{Q}_l$-adic étale sheaves (via étale homology; we call the heart of t_{MM} the category of mixed motives over K). Then t_{MM} also exists for motives over any 'reasonable' (see below) $\operatorname{Spec}\mathbb{Z}[\frac{1}{l}]$-scheme S. So, one may say that a certain $MM(S)$ exists in this case. The proof if quite easy (given the properties of Beilinson motives established in [CiD09]); we just apply a simple gluing argument. Actually, it is not necessary to fix l here: if for each K of characteristic p there exists a (motivic) t-structure for $DM_{gm}(K)$ that is strictly compatible (as above) with $\mathbb{Q}_{l'}$-adic étale (co)homology for any $l' \neq p$, then the motivic t-structure exists over any (reasonable) S (and it does not depend on l).

The second central result seems to be more interesting and complicated. We prove that certain 'weights' exist for mixed motives over any equicharacteristic scheme S; if $\operatorname{char} S = 0$ or if the weights are 'nice' over a universal K of the same characteristic, then these weights are 'nice' over S. This phrase requires a considerable amount of explanation, and we give it here.

The 'classical' approach for constructing weights for motives (originating from Beilinson) was to define a filtration for motives that would 'split' Chow motives into their components corresponding to single (co)homology groups (i.e would yield the so-called Chow-Kunneth decompositions). Since the existence of Chow-Kunneth decompositions is very much conjectural, it is no wonder that this approach has not yielded any significant (general) results up to this moment (to the knowledge of the author).

An alternative method for defining (certain) weights for motives was proposed and successfully implemented in [Bon10a]. To this end *weight structures* for triangulated categories were defined. This notion is a natural important counterpart of t-structures; somewhat similarly to t-structures, weight structures for a triangulated $\underline{C}$ are defined in terms of $\underline{C}_{w\le 0}, \underline{C}_{w\ge 0} \subset Obj\underline{C}$. For the Chow weight structure w_{Chow} for $DM_c(S)$ its *heart* $DM_c(S)_{w_{Chow}\le 0}\cap DM_c(S)_{w_{Chow}\ge 0}$ consists of Chow motives (over S; those are 'ordinary' Chow motives if S is the spectrum of a perfect field); we avoid Chow-Kunneth decompositions this way. w_{Chow} allows to define certain *(Chow)-weight filtrations* and *(Chow)-weight spectral sequences* for any (co)homology of motives; for singular and étale cohomology those are isomorphic to the 'classical' ones. w_{Chow} for $DM_c(S)$ was introduced in [Heb11] and [Bon10c]; it is closely related with the weights for mixed complexes of sheaves (as introduced in §5.1.8 of [BBD82]; see §§3.4–3.6 of [Bon10c]) and of mixed Hodge complexes and modules (see §2.3 of [Bon12]). All of these results are unconditional.

In [Bon12] and (especially) in the current paper we demonstrate that the Chow weight structure is also useful for the study of motivic conjectures. In particular, we prove (using the results of §3 of [Bon12]): if t_{MM} exists over an equicharacteristic scheme S, then Chow-weight spectral sequences yield a weight filtration for S-motivic sheaves; this filtration is strictly respected by morphisms of motives. Our argument relies on the degeneration at E_2 of Chow-weight spectral sequences for 'perverse étale homology'. We prove the latter result unconditionally. It also yields the existence of the Chow-weight filtration for perverse étale (co)homology of motives that is strictly restricted by ('motivic') morphisms; so it could be useful for itself.

Moreover, in (§1 of) ibid. also the conjectural relation of w_{Chow} with the motivic t-structure was axiomatized. The corresponding notion of *transversal* weight and t-structures was introduced, and several equivalent definitions of transversality were given. So, we actually prove: if over a universal domain K of characteristic p (that could be 0) t_{MM} exists and is transversal to w_{Chow}, then the same is true for $DM_c(S)$ for any 'reasonable' S (i.e. if S is separated of finite type over an regular excellent separated scheme of dimension ≤ 1) of characteristic p.

This 'triangulated' approach to weights (for mixed motives) has serious advantages over the (usual) 'abelian' version. First, it allows to combine the conjectural properties of mixed motives with unconditional results on the Chow weight structure (and on Chow-weight spectral sequences). We obtain some 'new' properties of mixed motivic sheaves this way; note that (by the virtue of our results) all of them follow from 'standard' motivic conjectures (cf. the discussion in §4.1 below). Besides, we obtain a description of weights for mixed motives whose only conjectural ingredient is the existence of t_{MM}.

Lastly note that the 'triangulated' approach allows us to apply a certain gluing argument (that heavily relies on §1.4 of [BBD82]) that does not seem to work in the context of filtered abelian categories.

Summarizing: we prove that if a certain list of standard (motivic) conjectures over algebraically closed fields hold, then the category of mixed motivic sheaves exists over any 'reasonable' scheme S; if S is an equicharacteristic scheme, we also obtain 'nice weights' for S-motivic sheaves. We also deduce (most of) the properties of this category that were conjectured by Beilinson and others, and prove some of their 'triangulated extensions'. Besides, we prove a certain 'motivic decomposition theorem', and calculate the Grothendieck group of mixed motivic sheaves.

Lastly, we note that the results of the current paper (as well as the results of [Bon10c]) only rely upon a certain 'axiomatics' of Beilinson motives (i.e. on a certain list of their properties; cf. Remark 3.2.2(1) below). It follows that our arguments could be applied to the study of other categories satisfying similar properties. A natural candidate here would be M. Saito's Hodge modules. Yet it seems that this setting has been already thoroughly studied by Saito himself (cf. Proposition 2.3.1(I) of [Bon12]); on the other hand, our methods could possibly yield certain simplifications for his arguments.

Now we list the contents of the paper. Some more detail can be found at the beginnings of sections.

§1 is dedicated to the recollection of certain homological algebra. We recall some basics of t-structures. We also remind the reader basic definitions and results on weight structures, weight filtrations and spectral sequences, as well as the notion of transversality of weight and t-structures (following [Bon10a] and [Bon12]). We also recall (mostly from §1.4 of [BBD82]) several basic results on gluing of t-structures and weight structures.

In §2 we recall the basic properties of S-motives (as defined and studied in [CiD09]) and the Chow weight structure for them (as introduced in [Heb11] and [Bon10c]). We also study weight spectral sequences for the 'perverse étale homology'. Those degenerate at E_2 if S is an equicharacteristic scheme; we conjecture that they degenerate for a general (reasonable) S also.

In §3 we define the motivic t-structure (when it exists) as the one that is (strictly) compatible with the perverse t-structure for complexes of $\mathbb{Q}_l$-adic sheaves. We prove that the motivic t-structure exists over S if it exists over (all) universal domains. We also deduce some simple consequences from the 'niceness' of the motivic t-structure (i.e. of its transversality with w_{Chow}). They enable us to prove: over an equicharacteristic scheme there exists a nice t_l if the same is true over some universal domain of the same characteristic.

In §4 we verify that the existence of a (nice) motivic t-structure and its independence from l follows from a certain list of (more or less) 'standard' mo-

tivic conjectures (over algebraically closed base fields). We also prove a certain 'motivic Decomposition Theorem' (modulo the conjectures mentioned). In particular, we characterize semi-simple (pure) motives over S in terms of those over its residue fields. This enables us to calculate $K_0(DM_c(S))$.

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Notation. $\underline{C}$ below will always denote some triangulated category. t will always denote a bounded t-structure, and w will be a bounded weight structure (the theory of weight structures was thoroughly studied in [Bon10a]; see also §1.1 below).

$D \subset Obj\underline{C}$ will be called extension-stable if for any distinguished triangle $A \to B \to C$ in $\underline{C}$ we have: $A, C \in D \implies B \in D$. Note that $\underline{C}^{t\le i}$, $\underline{C}^{t\ge i}$, $\underline{C}^{t=0}$ (see §1.2), $\underline{C}_{w\ge i}$, and $\underline{C}_{w\le i}$ (see §1.1) are extension-stable for any t, w and any $i \in \mathbb{Z}$.

For $D, E \subset Obj\underline{C}$ we will write $D \perp E$ if $\underline{C}(X, Y) = \{0\}$ for all $X \in D$, $Y \in E$.

A full subcategory $B \subset \underline{C}$ is called *Karoubi-closed* in $\underline{C}$ if B contains all $\underline{C}$-retracts of its objects For $B \subset \underline{C}$ we will call the subcategory of $\underline{C}$ whose objects are all retracts of objects of B (in $\underline{C}$) the *Karoubi-closure* of B in $\underline{C}$.

For a class of objects $C_i \in Obj\underline{C}$, $i \in I$, we will denote by $\langle C_i \rangle$ the smallest strictly full triangulated subcategory containing all C_i.

$\underline{A}$ will always be an abelian category. We will call a covariant (resp. contravariant) additive functor $H : \underline{C} \to \underline{A}$ *homological* (resp. *cohomological*) if it converts distinguished triangles into long exact sequences.

All morphisms and schemes below will be separated. S will usually be our base scheme. Often $j : U \to S$ will be an open immersion, and $i : Z \to S$ will be the complimentary closed embedding.

All the schemes below will always be of finite type over some (excellent separated) regular scheme S_0 of dimension lesser than or equal to 1; we will say that they are *reasonable*.

Below l will always be a prime number (as well as l'); we will usually assume l to be fixed. p will usually denote the characteristic of some scheme (so it is either a prime number or 0); usually $p \neq l$. We will say that p is the characteristic of S (only) if it is an equicharacteristic p scheme (so it is an $\operatorname{Spec}\mathbb{F}_p$-scheme if $p > 0$ and a $\operatorname{Spec}\mathbb{Q}$-one for $p = 0$).

Below we will identify a Zariski point (of a scheme S) with the spectrum

of its residue field (sometimes we will also make no distinction between the spectrum of a field and the field itself). $\mathcal{S}$ will denote the set of (Zariski) points of S. For $K \in \mathcal{S}$ we will denote the natural morphism $K \to S$ by j_K. We will call the dimension of the closure of K in S the dimension of K.

All the motives that we will consider in this paper will have rational coefficients (we will not mention the coefficients in the notation; this includes $Chow$ and DM_{gm}).

1 Preliminaries on triangulated categories, weight- and t-structures

In §1.1 we recall some basics on weight structures (as developed in [Bon10a]).

In §1.2 we recall the definition of a t-structure and introduce some notation.

In §1.3 we study weight spectral sequences (following §2 of [Bon10a] and §3 of [Bon12]), their degeneration, and weight filtrations for $\underline{Ht}$ coming from w.

In §1.4 we recall the notion of transversal weight and t-structures (as introduced in [Bon12]).

In §1.5 we prove (heavily relying upon §1.4 of [BBD82]) several auxiliary statements on t-structures and weights in the 'gluing situation'.

1.1 Weight structures: short reminder

Definition 1.1.1. I A pair of subclasses $\underline{C}_{w\le 0}, \underline{C}_{w\ge 0} \subset Obj\underline{C}$ will be said to define a weight structure w for $\underline{C}$ if they satisfy the following conditions:

(i) $\underline{C}_{w\ge 0}, \underline{C}_{w\le 0}$ are additive and Karoubi-closed in $\underline{C}$ (i.e. contain all $\underline{C}$-retracts of their objects).

(ii) **Semi-invariance with respect to translations.**

$\underline{C}_{w\le 0} \subset \underline{C}_{w\le 0}[1]$, $\underline{C}_{w\ge 0}[1] \subset \underline{C}_{w\ge 0}$.

(iii) **Orthogonality.**

$\underline{C}_{w\le 0} \perp \underline{C}_{w\ge 0}[1]$.

(iv) **Weight decompositions**.

For any $M \in Obj\underline{C}$ there exists a distinguished triangle

$$B \to M \to A \xrightarrow{f} B[1] \quad (1)$$

such that $A \in \underline{C}_{w\ge 0}[1]$, $B \in \underline{C}_{w\le 0}$.

II The category $\underline{Hw} \subset \underline{C}$ whose objects are $\underline{C}_{w=0} = \underline{C}_{w\ge 0} \cap \underline{C}_{w\le 0}$, $\underline{Hw}(Z,T) = \underline{C}(Z,T)$ for $Z, T \in \underline{C}_{w=0}$, will be called the *heart* of w.

III $\underline{C}_{w\ge i}$ (resp. $\underline{C}_{w\le i}$, resp. $\underline{C}_{w=i}$) will denote $\underline{C}_{w\ge 0}[i]$ (resp. $\underline{C}_{w\le 0}[i]$, resp. $\underline{C}_{w=0}[i]$).

IV We will say that $(\underline{C}, w)$ is *bounded* if $\cup_{i\in\mathbb{Z}}\underline{C}_{w\le i} = Obj\underline{C} = \cup_{i\in\mathbb{Z}}\underline{C}_{w\ge i}$.

V Let $\underline{C}$ and $\underline{C}'$ will be triangulated categories endowed with weight structures w and w', respectively; let $F : \underline{C} \to \underline{C}'$ be an exact functor.

F will be called *left weight-exact* (with respect to w, w') if it maps $\underline{C}_{w\le 0}$ to $\underline{C}'_{w'\le 0}$; it will be called *right weight-exact* if it maps $\underline{C}_{w\ge 0}$ to $\underline{C}'_{w'\ge 0}$. F is called *weight-exact* if it is both left and right weight-exact.

Remark 1.1.2. 1. A simple (and yet useful) example of a weight structure comes from the stupid filtration on the homotopy categories $K(B) \supset K^b(B)$ of cohomological complexes for an arbitrary additive category B. In this case $K(B)_{w\le 0}$ (resp. $K(B)_{w\ge 0}$) will be the class of complexes that are homotopy equivalent to complexes concentrated in degrees ≥ 0 (resp. ≤ 0). The heart of this weight structure (either for $K(B)$ or for $K^b(B)$) is the the Karoubi-closure of B in the corresponding category.

2. A weight decomposition (of any $M \in Obj\underline{C}$) is (almost) never canonical. Yet for an $m \in \mathbb{Z}$ we will often need an (arbitrary) choice of a weight decomposition of $X[-m]$ shifted by $[m]$. This way we obtain a distinguished triangle

$$w_{\le m}X \to X \to w_{\ge m+1}X \qquad (2)$$

with some $w_{\ge m+1}X \in \underline{C}_{w\ge m+1}$, $w_{\le m}X \in \underline{C}_{w\le m}$ (see Remark 1.2.2 of [Bon10a]); we will use this notation below (though $w_{\ge m+1}X$ and $w_{\le m}X$ are not uniquely determined by X, unless we impose some additional restrictions on these objects).

3.**Caution on signs of weights.** When the author defined weight structures (in [Bon10a]), he considered $(\underline{C}^{w\le 0}, \underline{C}^{w\ge 0})$ such that $\underline{C}^{w\le 0}$ is stable with respect to [1] (similarly to the usual convention for t-structures); in particular, this meant that for $\underline{C} = K(B)$ and for the 'stupid' weight structure for it mentioned above a complex C whose only non-zero term is the fifth one (i.e. $C^5 \ne 0$) was 'of weight 5'. Whereas this convention seems to be quite natural, for weights of mixed Hodge complexes, mixed Hodge modules (see Proposition 2.6 of [Bon12]), and mixed complexes of sheaves (see Proposition 3.6.1 of [Bon10c]) 'classically' exactly the opposite convention was used (by Beilinson, Saito and others; so, in this convention our C is of weight -5). For this reason, in the current paper we use the 'reverse' convention for signs of weights, that is compatible with the 'classical weights' (this convention for the Chow weight structure for motives was used in [Heb11], in [Bon12], and in [Bon10c]); so the signs of weights used below will be opposite to those in [Bon10a] and in [Bon10b].

Now we recall those properties of weight structures that will be needed below.

Proposition 1.1.3. *Let $\underline{C}$ be a triangulated category. w will be a weight structure for $\underline{C}$ everywhere except assertion 1.*

1. *(C_1, C_2) $(C_1, C_2 \subset Obj\underline{C})$ define a weight structure for $\underline{C}$ whenever (C_2^{op}, C_1^{op}) define a weight structure for $\underline{C}^{op}$.*

2. *$\underline{C}_{w\le 0}$, $\underline{C}_{w\ge 0}$, and $\underline{C}_{w=0}$ are extension-stable.*

3. *Suppose that v is another weight structure for $\underline{C}$; let $\underline{C}_{v\le 0} \subset \underline{C}_{w\le 0}$ and $\underline{C}_{v\ge 0} \subset \underline{C}_{w\ge 0}$. Then $v = w$ (i.e. the inclusions are equalities).*

4. *If w is bounded, then $\underline{C}_{w\le 0}$ is the smallest extension-stable subclass of $Obj\underline{C}$ containing $\cup_{i\le 0}\underline{C}_{w=i}$; $\underline{C}_{w\ge 0}$ is the smallest extension-stable class of $Obj\underline{C}$ containing $\cup_{i\ge 0}\underline{C}_{w=i}$.*

Proof. All of the assertions were proved in [Bon10a]; see Propositions 1.3.3, 1.5.6, and 2.1.2 of ibid (keeping in mind Remark 1.1.2(3)!). ☐

1.2 t-structures: a very short reminder and notation

To fix the notation we recall the definition of a t-structure.

Definition 1.2.1. A pair of subclasses $\underline{C}^{t\ge 0}, \underline{C}^{t\le 0} \subset Obj\underline{C}$ will be said to define a t-structure t if they satisfy the following conditions:

(i) $\underline{C}^{t\ge 0}, \underline{C}^{t\le 0}$ are strict i.e. contain all objects of $\underline{C}$ isomorphic to their elements.

(ii) $\underline{C}^{t\ge 0} \subset \underline{C}^{t\ge 0}[1]$, $\underline{C}^{t\le 0}[1] \subset \underline{C}^{t\le 0}$.

(iii) **Orthogonality**. $\underline{C}^{t\le 0}[1] \perp \underline{C}^{t\ge 0}$.

(iv) t-**decompositions**.

For any $X \in Obj\underline{C}$ there exists a distinguished triangle

$$A \to X \to B \to A[1] \tag{3}$$

such that $A \in \underline{C}^{t\le 0}, B \in \underline{C}^{t\ge 0}[-1]$.

Bounded t-structures can be defined similarly to Definition 1.1.1(IV). All bounded t-structures are *non-degenerate* i.e. $\cap_{i\in\mathbb{Z}}\underline{C}^{t\le i} = \cap_{i\in\mathbb{Z}}\underline{C}^{t\ge i} = \{0\}$; this is equivalent to the fact the the collection of functors H_i^t, $i \in \mathbb{Z}$ (that we will define now) is conservative.

We will need some more notation for t-structures.

Definition 1.2.2. 1. The category $\underline{Ht}$ whose objects are $\underline{C}^{t=0} = \underline{C}^{t\geq 0} \cap \underline{C}^{t\leq 0}$, $\underline{Ht}(X,Y) = \underline{C}(X,Y)$ for $X, Y \in \underline{C}^{t=0}$, will be called the *heart* of t. Recall that $\underline{Ht}$ is always abelian; short exact sequences in $\underline{Ht}$ come from distinguished triangles in $\underline{C}$.

2. $\underline{C}^{t\geq l}$ (resp. $\underline{C}^{t\leq l}$) will denote $\underline{C}^{t\geq 0}[-l]$ (resp. $\underline{C}^{t\leq 0}[-l]$).

Remark 1.2.3. 1. Recall that (3) defines additive functors $\underline{C} \to \underline{C}^{t\leq 0} : X \to A$ and $C \to \underline{C}^{t\geq 1} : X \to B$. We will denote A, B by $X^{\tau\leq 0}$ and $X^{\tau\geq 1}[-1]$, respectively. (3) will be called the *t-decomposition* of X.

More generally, the t-components of $X[i]$ (for any $i \in \mathbb{Z}$) will be denoted by $X^{\tau\leq i} \in \underline{C}^{t\leq 0}$ and $X^{\tau\geq i+1}[-1] \in \underline{C}^{t\geq 1}$, respectively.

$\tau_{\leq i}X$ will denote $X^{\tau\leq i}[-i]$; $\tau_{\geq i}X$ will denote $X^{\tau\geq i}[-i]$.

2. The functor $X \mapsto \tau_{\geq 0}X$ is left adjoint to the inclusion $\underline{C}^{t\geq 0} \to \underline{C}$.

3. We will also need the following easy (and well-known) properties of t-structures.

The first one is Proposition 1.3.17(iii) of [BBD82]: if a functor F is left adjoint to G, and their targets are endowed with t-structures, then F is right t-exact whenever G is left t-exact. The latter assertions mean that F it respects 't-negative' objects, whereas G respects t-positive ones.

The second property is: if for two t-structures t and t' on a triangulated $\underline{C}$ the identity functor is t-exact (for the pairs $(\underline{C}, t)$ and $(\underline{C}, t')$, i.e. $\underline{C}^{t\leq 0} \subset \underline{C}^{t'\leq 0}$ and $\underline{C}^{t\geq 0} \subset \underline{C}^{t'\geq 0}$), then $t = t'$. Indeed, the previous statement yields that the identity is also t-exact as a functor from $(\underline{C}, t')$ to $(\underline{C}, t)$.

We denote by H_0^t the zeroth homology functor corresponding to t. Shifting the t-decomposition of $X^{\tau\leq 0}[-1]$ by [1] we obtain a canonical and functorial (with respect to X) distinguished triangle $\tau_{\leq -1}X \to \tau_{\leq 0}X \to H_0^t(X)$; we denote $H_0^t(X[i])$ by $H_i^t(X)$.

1.3 On weight filtrations and (degenerating) weight spectral sequences

Now we recall certain properties of weight filtrations and weight spectral sequences. Most of them were established in §2 of [Bon10a], whereas the degeneration of weight spectral sequences was studied in §3 of [Bon12].

Let $\underline{A}$ be an abelian category. In §2 of [Bon10a] for $H : \underline{C} \to \underline{A}$ that is either cohomological or homological (i.e. it is either covariant or contravariant, and converts distinguished triangles into long exact sequences) certain *weight filtrations* and *weight spectral sequences* (corresponding to w) were introduced. Below we will be more interested in the homological functor case; certainly, one can pass to cohomology by a simple reversion of arrows (cf. §2.4 of ibid.).

Definition 1.3.1. Let $H : \underline{C} \to \underline{A}$ be a covariant functor, $i \in \mathbb{Z}$.

1. We denote $H \circ [i] : \underline{C} \to \underline{A}$ by H_i.

2. We choose some $w_{\le i}X$ and define the *weight filtration* for H by $W_iH : X \mapsto \operatorname{Im}(H(w_{\le i}X) \to H(X))$.

Recall that W_iH is functorial in X (in particular, it does not depend on the choice of $w_{\le i}X$); see Proposition 2.1.2(1) of ibid.

Now we recall some of the properties of weight spectral sequences; we are especially interested in the case when they degenerate.

Proposition 1.3.2. *I For a homological X and any $X \in Obj\underline{C}$ there exists a spectral sequence $T = T_w(H, X)$ with $E_1^{pq}(T) = H_q(X^p)$ for certain $X^m \in \underline{C}_{w=0}$ (coming from certain weight decompositions as in (2)) that converges to $E_\infty^{p+q} = H_{p+q}(X)$. T is $\underline{C}$-functorial in X and in H (with respect to composition of H with exact functors of abelian categories) starting from E_2. Besides, the step of filtration given by $(E_\infty^{l,m-l} : l \ge k)$ on $H_m(X)$ (for some $k, m \in \mathbb{Z}$) equals $(W_{-k}H_m)(X)$. Moreover, $T(H, X)$ comes from an exact couple with $D_1^{pq} = H_{p+q}(w_{\le -p}X)$ (here one can fix any choice of $w_{\le -p}X$).*

We will say that T degenerates at E_2 (for a fixed H) if $T_w(H, X)$ does so for any $X \in Obj\underline{C}$.

II Suppose that T degenerates at E_2 (as above), $i \in \mathbb{Z}$. Then the following statements are fulfilled.

1. The functors W_iH and $W_i'H : X \mapsto H(X)/W_{i-1}H(X)$ are homological.

2. For any $f \in \underline{C}(X, Y)$ the morphism $H(f)$ is strictly compatible with the filtration of H by W_i i.e. $W_iH(X)$ surjects onto $W_iH(Y) \cap \operatorname{Im} H(f)$.

Proof. Immediate from Proposition 3.1.2 of [Bon12].

□

Now we introduce the notion of a weight filtration for an abelian category following Definition E7.2 of [B-VK10].

Definition 1.3.3. For an abelian $\underline{A}$, we will say that an increasing family of full subcategories $\underline{A}_{\le i} \subset \underline{A}$, $i \in \mathbb{Z}$, yield a *weight filtration* for $\underline{A}$ if $\cap_{i \in \mathbb{Z}} \underline{A}_{\le i} = \{0\}$, $\cup_{i \in \mathbb{Z}} \underline{A}_{\le i} = \underline{A}$, and there exist exact right adjoints $W_{\le i}$ to the embeddings $\underline{A}_{\le i} \to \underline{A}$.

We will need the following statement.

Lemma 1.3.4. *Let $\underline{A}_{\le m}$, $m \in \mathbb{Z}$ yield a weight filtration for $\underline{A}$. Then the following statements are valid.*

1. $\underline{A}_{\le m}$ are are exact abelian subcategories of $\underline{A}$.

2. All $W_{\leq m}$ are idempotent functors.

3. The adjunctions yield functorial embeddings of $W_{\leq m}X \to X$ such that $W_{\leq m-1}X \subset W_{\leq m}X$ for all $m \in \mathbb{Z}$, and the functors $W_{\geq m} : X \mapsto X/W_{\leq m-1}X$ are exact also.

4. The categories $\underline{A}_m$ being the 'kernels' of the restriction of $W_{\leq m-1}$ to $\underline{A}_{\leq m}$, are abelian, and $\underline{A}_m \perp \underline{A}_j$ for any $j \neq m$.

Proof. This is (a part of) Lemma 2.1.2 of [Bon12].

□

Now we fix certain (bounded) w and t for $\underline{C}$, and study a condition ensuring that w induces a weight filtration for $\underline{Ht}$.

Proposition 1.3.5. *Let $H = H_0^t$.*

I Suppose that the corresponding T degenerates. Then the functors $W_iH : \underline{C} \to \underline{Ht}$ are homological. The restrictions $W_{\leq i}$ of W_iH to $\underline{Ht}$ define a weight filtration for this category. Besides, $W_iH = W_{\leq i} \circ H$.

II Let $\underline{B}$ be an abelian category; let $F : \underline{Ht} \to \underline{B}$ be an exact functor.

1. Suppose that T degenerates. Then $T_w(F \circ H, -)$ also does.

2. Conversely, suppose that F is conservative and that $T_w(F \circ H, -)$ degenerates. Then T degenerates.

Moreover, for $X \in \underline{C}^{t=0}$ we have: $W_{\leq i}X = X$ (resp. $W_{\leq i}X = 0$) whenever $W_i(F \circ H)(X) = F(X)$ (resp. $W_i(F \circ H)(X) = 0$).

Proof. This is (a part of) Proposition 3.2.1 of [Bon12].

□

1.4 On transversal weight and t-structures

Let t be a t-structure for $\underline{C}$, and w be a weight structure for it.

Definition 1.4.1. 1. For some $\underline{C}, t, w$ we will say that a distinguished triangle (2) (for some m, X) is *nice* if $w_{\leq m}X, X, w_{\geq m+1}X \in \underline{C}^{t=0}$.

We will also say that this distinguished triangle is a *nice decomposition* of X (for the corresponding m).

2. Let t and w be bounded.

We will say that t and w are transversal if a nice decomposition exists for any $m \in \mathbb{Z}$ and any $X \in \underline{C}^{t=0}$.

Proposition 1.4.2. *I We fix some $\underline{C}, w, t, m$; suppose that for a certain $N \subset \underline{C}^{t=0}$ a nice decomposition exists for any $X \in N$. Consider $N' \subset \underline{C}^{t=0}$ being the smallest subclass containing N that satisfies the following condition: if $A, C \in N'$,*

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{4}$$

is a complex (i.e. $g \circ f = 0$), f is monomorphic, g is epimorphic, $\operatorname{Ker} g / \operatorname{Im} f \in N'$, *then* $B \in N'$. *Then a nice choice of (2) exists for any* $X \in N'$.

II If t is transversal to w, then the following statements are fulfilled for any $i \in \mathbb{Z}$, $X \in Obj\underline{C}$, $Y \in \underline{C}^{t=0}$.

1. For any H that could be presented as $F \circ H_m^t$, where $F : \underline{Ht} \to \underline{A}$ is an exact functor, $T_w(H, -)$ degenerates at E_2.

2. Nice decompositions exist and are $\underline{Ht}$-functorial in X (for a fixed m). The corresponding functor $W_{\le m} : X \mapsto w_{\le m}X$ can be described as (the restriction to $\underline{Ht}$ of) $W_m H_0^t$ (see Definition 1.3.1(2)); i.e. it coincides with the functor $W_{\le m}$ given by Proposition 1.3.5(I).

3. The category $\underline{A}_m = \underline{C}^{t=0} \cap \underline{C}_{w=m}$ is (abelian) semi-simple; there is a splitting $\underline{C}_{w=0} = \bigoplus_{m \in \mathbb{Z}} Obj\underline{A}_m[-m]$ given by $Y \mapsto \bigoplus_{i \in \mathbb{Z}} H_i^t(Y)[-i]$.

4. $W_{\le m}X$ yield an increasing filtration for X whose m-th factor belongs to $\underline{A}_m$. Moreover, this filtration is uniquely and functorially determined by this condition.

5. $Y \in \underline{C}_{w \le m}$ (resp. $Y \in \underline{C}_{w \ge m}$) whenever for any $j \in \mathbb{Z}$ we have $W_{\le m+j}(H_j^t(Y)) = H_j^t(Y)$ (resp. $W_{\le m+j-1}(H_j^t(Y)) = 0$).

III For any t, w, $m \in \mathbb{Z}$, any $X \in \underline{C}^{t=0}$, and any choice of $w_{\le m}X$ (and a morphism $w_{\le m}X \to X$ corresponding to a weight decomposition) consider the morphism $f_m(X) : (W_m H_0^t)(X) \to X$ (cf. Definition 1.3.1(2)). Then t is transversal to w whenever this morphism extends to a nice decomposition (for any X, m).

IV For a family of semi-simple (abelian) $\{\underline{A}_m \subset \underline{C},\ m \in \mathbb{Z}\}$, suppose that $\langle \cup_{m \in \mathbb{Z}} Obj\underline{A}_m \rangle = \underline{C}$, and $\underline{A}_m \perp \underline{A}_j[s]$ for any $m, j, s \in \mathbb{Z}$ such that: either $s < 0$, or $s > m - j$, or $s = 0$ and $m > j$.

Then there exist transversal w and t such that $\underline{C}_{w=0} = \bigoplus_m Obj\underline{A}_m[-m]$, and $\underline{Ht}$ is the smallest extension-stable subcategory of $\underline{C}$ containing $\cup \underline{A}_m$.

V For any t, w, t is transversal to w whenever there exists a family of semi-simple $\underline{A}_m \subset \underline{Ht}$ ($m \in \mathbb{Z}$) such that: $Obj\underline{A}_m \cap Obj\underline{A}_j = \{0\}$ for all $j \neq m$, $j \in \mathbb{Z}$, and $\underline{C}_{w=0} = \bigoplus_m Obj\underline{A}_m[-m]$.

Proof. I This is Lemma 1.1.3 of [Bon12].

II 1. Immediate from Proposition 3.2.1(II,III1) of ibid.

2. The functoriality of nice decompositions is the condition (iv') of Theorem 1.2.1 of ibid. (which is equivalent to condition (iv) of loc. cit. that we took above for the definition of transversality). The equality of two distinct descriptions of $W_{\le m}$ is given by Proposition 3.2.1(II) of ibid.

3. See Remark 1.2.3(2) of [Bon12].

4. Immediate from condition (iii') of Theorem 1.2.1 of ibid. (cf. the proof of assertion II2).

5. This is Proposition 1.2.4(I2) of ibid.

III See Remark 1.2.3(4) of ibid.

IV Theorem 1.2.1 of ibid. implies that such a family of $\underline{A}_m$ does yield some transversal structures t, w. Besides, loc. cit. also allows to calculate $\underline{Ht}$, whereas $\underline{C}_{w=0} = \bigoplus_m Obj\underline{A}_m[-m]$ by Remark 1.8(2) of ibid.

V Since $\langle \underline{Hw} \rangle = \underline{C}$, we obtain that $\langle \cup_{m\in\mathbb{Z}} Obj\underline{A}_m \rangle = \underline{C}$. Since $\underline{A}_m$ are semi-simple, we obtain that $\underline{A}_m \perp \underline{A}_j$ for any $m \neq j$. The orthogonality axioms of weight and t-structures also yield the remaining orthogonality conditions that are needed in order to apply the previous assertion. We obtain that certain transversal t' and w' exist; besides, $\underline{Hw'} = \underline{Hw}$ and $\underline{Ht'} \subset \underline{Ht}$. Since w' is bounded, this implies $w = w'$ (see Proposition 1.1.3(4,3)). Since t' is bounded also, we easily deduce that $t' = t$.

□

Remark 1.4.3. In the case of motives (of smooth projective varieties over a field) the splittings mentioned in assertions II3 and IV corresponds to the so-called *Chow-Kunneth* decompositions ('of the diagonal').

1.5 Some auxiliary 'gluing statements'

Below we will apply several gluing arguments. We chose to gather the definitions and auxiliary statements related with this matter here.

Definition 1.5.1. 1. The set $(\underline{C}, \underline{D}, \underline{E}, i_*, j^*, i^*, i^!, j_!, j_*)$ is called *gluing data* if it satisfies the following conditions.

(i) $\underline{C}, \underline{D}, \underline{E}$ are triangulated categories; $i_* : \underline{D} \to \underline{C}$, $j^* : \underline{C} \to \underline{E}$, $i^* : \underline{C} \to \underline{D}$, $i^! : \underline{C} \to \underline{D}$, $j_* : \underline{E} \to \underline{C}$, $j_! : \underline{E} \to \underline{C}$ are exact functors.

(ii) i^* (resp. $i^!$) is left (resp. right) adjoint to i_*; $j_!$ (resp. j_*) is left (resp. right) adjoint to j^*.

(iii) i_* is a full embedding; j^* is isomorphic to the localization (functor) of $\underline{C}$ by $i_*(\underline{D})$.

(iv) For any $X \in Obj\underline{C}$ the pairs of morphisms $j_!j^*X \to X \to i_*i^*X$ and $i_*i^!X \to X \to j_*j^*X$ can be completed to distinguished triangles (here the connecting morphisms come from the adjunctions of (ii)).

(v) $i^*j_! = 0$; $i^!j_* = 0$.

(vi) All of the adjunction transformations $i^*i_* \to id_{\underline{D}} \to i^!i_*$ and $j^*j_* \to id_{\underline{E}} \to j^*j_!$ are isomorphisms of functors.

2. In the setting of part 1 of this definition, we will say that $X \in Obj\underline{C}$ is a *lift* of an $Y \in Obj\underline{E}$ if $j^*X \cong Y$. Similarly, for a lift of a distinguished triangle C in $\underline{E}$ is a distinguished triangle C' in $\underline{C}$ such that $j^*C' \cong C$.

3. In the setting of part 1, suppose that $\underline{C}$ is endowed with a t-structure $t = t_{\underline{C}}$. We define the *intermediate image* functor $j_{!*} : \underline{E} \to \underline{C}$ as $X \mapsto$

$\mathrm{Im}(H_0^{t_{\underline{C}}} j_! X \to H_0^{t_{\underline{C}}} j_* X)$; here the morphism $j_! \to j_*$ comes from adjunction (and we use the fact that $j^* j_! \cong j^* j_! \cong 1_{\underline{E}}$; cf. (1.4.6.2) and Definition 1.4.22 of [BBD82]).

4. In the setting of part 1, suppose also that $\underline{D}$ and $\underline{E}$ are endowed with certain t-structures $t_{\underline{D}}$ and $t_{\underline{E}}$, respectively. Then we will say that a t-structure $t = t_{\underline{C}}$ for $\underline{C}$ is *glued from* $t_{\underline{D}}$ and $t_{\underline{E}}$ if we have: $\underline{C}^{t_{\underline{C}} \le 0} = \{X \in Obj\underline{C} : j^* X \in \underline{E}^{t_{\underline{E}} \le 0}$ and $i^* X \in \underline{D}^{t_{\underline{D}} \le 0}\}$, and $\underline{C}^{t_{\underline{C}} \ge 0} = \{X \in Obj\underline{C} : j^* X \in \underline{E}^{t_{\underline{E}} \ge 0}$ and $i^! X \in \underline{D}^{t_{\underline{D}} \ge 0}\}$. In this case we will also say that $\underline{C}$, $\underline{D}$, and $\underline{E}$ are *endowed with compatible t-structures.*

Remark 1.5.2. Our definition of the gluing data is far from being the 'minimal' one. Actually, it is well known (see Chapter 9 of [Nee01]) that a gluing data can be uniquely recovered from an inclusion $\underline{D} \to \underline{C}$ of triangulated categories that admits both a left and a right adjoint functor.

Our notation for the connecting functors is (certainly) coherent with Proposition 2.1.1 below.

Proposition 1.5.3. *I In the setting of Definition 1.5.1(1) assume that $\underline{D}$ is endowed with a t-structure $t_{\underline{D}}$. Then for any $X \in Obj\underline{C}$ any distinguished triangle $A' \to X'(= j^* X) \to B'$ (in $\underline{E}$) possesses a lift $A \to X \to B$ (see Definition 1.5.1(2)) such that $i^* A \in \underline{D}^{t_{\underline{D}} \le 0}$, and $i^! B \in \underline{D}^{t_{\underline{D}} \ge 1}$.*

II In the setting of Definition 1.5.1(4) the following statements are fulfilled.

1. There exists a t-structure $t_{\underline{C}}$ for $\underline{C}$ glued from $t_{\underline{D}}$ and $t_{\underline{E}}$.

2. $t_{\underline{C}}$ is characterized by the following property: i_* and j^* are t-exact.

Moreover, $j_!$ and i^* are right t-exact (see Remark 1.2.3(3)), whereas j_* and $i^!$ are left t-exact (with respect to $t_{\underline{D}}$, $t_{\underline{C}}$, and $t_{\underline{E}}$, respectively).

III In the setting of Definition 1.5.1(1) assume that $\underline{C}, \underline{D}, \underline{E}$ are endowed with weight structures $w_{\underline{C}}$, $w_{\underline{D}}$, and $w_{\underline{E}}$, respectively, and that i_* and j^* are weight-exact. Then we will say that $w_{\underline{C}}$, $w_{\underline{D}}$, and $w_{\underline{E}}$ are compatible.

In this situation $j_!$ and i^* are left weight-exact, whereas j_* and $i^!$ are right weight-exact. Besides, we have: $\underline{C}_{w \ge 0} = \{M \in Obj\underline{C} :\ i^!(M) \in \underline{D}_{w_{\underline{D}} \ge 0},\ j^*(M) \in \underline{E}_{w_{\underline{E}} \ge 0}\}$ and $\underline{C}_{w \le 0} = \{M \in Obj\underline{C} :\ i^*(M) \in \underline{D}_{w_{\underline{D}} \le 0},\ j^*(M) \in \underline{E}_{w_{\underline{E}} \le 0}\}$.

IV Assume that $\underline{C}$, $\underline{D}$, and $\underline{E}$ are endowed with compatible t-structures (see Definition 1.5.1(4)). Then for any $X, Y \in \underline{C}^{t=0}$, $X', Y' \in \underline{E}^{t=0}$, $i \in \mathbb{Z}$ the following statements are valid.

1. $j^* j_! X' \cong j^* j_{!*} X' \cong j^* j_* X' \cong X'$.

2. $i^* j_{!*} X' \cong \tau_{\le -1} i^* j_* X'$, *and* $i^! j_{!*} X' \cong \tau_{\ge 1} i^! j_! X'$.

3. *If a complex $A \to B \to C$ (in $\underline{Ht}_{\underline{D}}$) is exact in the term B, then the middle-term homology object of the complex $j_{!*}A \to j_{!*}B \to j_{!*}C$ belongs to $i_*\underline{D}^{t=0}$.*

4. *X can be obtained from $j_{!*}j^*X$ via two extensions by elements of $i_*\underline{D}^{t=0}$.*

5. *$j_{!*}$ maps monomorphisms to monomorphisms, and epimorphisms to epimorphisms.*

6. *$j_{!*}X'$ does not have non-trivial subobjects of factor-objects belonging to $i_*\underline{Ht}_{\underline{D}}$.*

7. *The homomorphism $\underline{C}(j_{!*}X', j_{!*}Y') \to \underline{E}(X', Y')$ induced by j^* is bijective.*

8. *If X' is simple, $j_{!*}X'$ also is.*

9. *If X' is semi-simple, then $j_{!*}X'$ can be functorially characterized as a semi-simple lift of X' none of whose components are killed by j^*.*

Proof. I We argue as in the proof of Theorem 1.4.10 of [BBD82]. We consider $Y = \mathrm{Cone}(X \to j_*B')[-1]$ and $A = \mathrm{Cone}(Y \to i_*(\tau_{\underline{D}, \geq 1} i^*Y))[-1]$. We complete the commutative triangle $A \to Y \to X$ to an octahedral diagram

$$\begin{tikzcd}
j_*B' \arrow[dr, "{[1]}"] \arrow[dd, "{[1]}"'] & & X \arrow[ll] \\
& Y \arrow[ur] \arrow[dl] & \\
i_*\tau_{\underline{D}, \geq 1} i^*Y \arrow[rr, "{[1]}"] & & A \arrow[uu] \arrow[ul]
\end{tikzcd}$$

and denote its sixth vertex by B.

Now we argue exactly as in loc. cit. (using the fact that exact functors convert distinguished triangles into distinguished ones, and the 'axioms' of gluing data). We obtain that $j^*(i_*\tau_{\underline{D}, \geq 1} i^*Y \to B \to j_*B') \cong (0 \to j^*B \to B')$; hence $j^*B \cong B'$. Next, $j^*(A \to X \to B) \cong (j^*A \to X' \to B')$; hence $j^*A \cong A'$. Furthermore, $i^*(A \to Y \to i_*\tau_{\underline{D}, \geq 1} i^*Y) \cong (i^*A \to i^*Y \to \tau_{\underline{D}, \geq 1} i^*Y)$; hence $i^*A \cong \tau_{\underline{D}, \leq 0} i^*Y$. It remains to note that $i^!(i_*\tau_{\underline{D}, \geq 1} i^*Y \to B \to j_*B') \cong (\tau_{\underline{D}, \geq 1} i^*Y \to i^*B \to 0)$; hence $i^!B \cong \tau_{\underline{D}, \geq 1} i^*Y$.

II1. This is (exactly) Theorem 1.4.10 of [BBD82].

II2. Obviously, if t is glued from $t_{\underline{D}}$ and $t_{\underline{E}}$, then j^* is t-exact. Since $j^*i_* = 0$, the adjunctions to i_* also yield that i_* is t-exact; see Remark 1.2.3(3).

Now, suppose that the t-exactness of i_* and j^* is fulfilled for some t-structure t' for $\underline{C}$. Then loc. cit. yields all of our t-exactness statements for t' (and so, they are fulfilled for t). It follows that $1_{\underline{C}}$ is t-exact as a functor from $(\underline{C}, t')$ to $(\underline{C}, t)$. Applying the other statement in loc. cit., we obtain that $t = t'$.

III Immediate from Proposition 1.2.3(13,15) of [Bon10c].

IV The proofs are easy applications of the results of (the end of) §1.4 of [BBD82].

(IV1) is immediate from the adjunctions and the t-exactness of i_*.

(IV2) is immediate from Proposition 1.4.23 of ibid.

(IV3): The previous assertion yields that the middle term homology in question is killed by j^*. Since the categorical kernel of j^* is $i_*\underline{D}$, and i_* is t-exact, we obtain the result.

(IV4): By assertion IV1, we have a $\underline{Ht}_{\underline{C}}$-epimorphism $a : H_0^{t_{\underline{C}}} j_! X \to j_{!*}X$, and a $\underline{Ht}_{\underline{C}}$-monomorphism $b : j_{!*}X \to H_0^{t_{\underline{C}}} j_* X$; both of them become isomorphisms after the application of j^*. Besides, adjunctions yield that $b \circ a$ factorizes through X. As in the proof of (IV3), the result follows immediately.

(IV5,6): Immediate from Corollary 1.4.25 of ibid.

(IV7) is an easy consequence of (IV6). Indeed, since $j^* j_{!*} \cong 1_{\underline{Ht}_{\underline{E}}}$, it suffices to verify that the homomorphism $\underline{C}(j_{!*}X', j_{!*}Y') \to \underline{E}(X', Y')$ induced by j^* is injective. Let $f \in \underline{C}(j_{!*}X', j_{!*}Y')$, $f \neq 0$. Then assertion (IV6) yields that $\operatorname{Im} f$ is not isomorphic to an object of $i_*(\underline{Ht}_{\underline{D}})$. Hence $j^* \operatorname{Im} f \neq 0$; since j^* is t-exact we obtain that $j^* f \neq 0$.

(IV8): This is just Proposition 1.4.26 of ibid.

(IV9): We may assume that X' is simple. Then $j_{!*}X$ is simple also by the previous assertion. Assertion IV4 yields that $j_{!*}X$ is the only simple lift of X'. Lastly, assertion IV6 implies that this characterization of $j_{!*}X$ is functorial.

$\square$

Remark 1.5.4. So, $j_{!*}X'$ is the 'minimal' lift of X'. As a consequence, when we will 'lift nice decompositions' (in the proof of Theorem 3.4.1 below) it will be sufficient to check whether $j_{!*}$ 'respects weights'. In order to verify the latter assertion, we will apply Theorem 2.3.3(II).

2 Recollection of relative motives and $\mathbb{Q}_l$-sheaves

In §2.1 we recall some of basic properties of Beilinson motives over S (as defined in [CiD09]).

In §2.2 we recall certain properties of the Chow weight structure w_{Chow} for $DM_c(S)$ (as introduced in [Heb11] and [Bon10c]).

In §2.3 we study weights for mixed sheaves and relate them with (the degeneration of) Chow-weight spectral sequences for $H^{et}_{\mathbb{Q}_l,0}$. The latter degenerate at E_2 if S is an equicharacteristic scheme (we conjecture that they degenerate for a general reasonable S also). This yields that the Chow-weight filtration for such (co)homology is strictly restricted by ('motivic') morphisms.

2.1 Beilinson S-motives (after Cisinski and Deglise)

We list some of the properties of the triangulated categories of Beilinson motives (this is the version of relative Voevodsky's motives with rational coefficients described by Cisinski and Deglise).

Proposition 2.1.1. *Let X, Y be any (reasonable) schemes; $f : X \to Y$ is a (separated) finite type morphism.*

1. *A tensor triangulated $\mathbb{Q}$-linear category $DM_c(X)$ with the unit object $\mathbb{Q}_X$ is defined.*

   $DM_c(X)$ is the category of constructible Beilinson motives over X, as defined (and thoroughly studied) in §14 of [CiD09].

2. *If S is the spectrum of a perfect field, $DM_c(S)$ is isomorphic to the category $DM_{gm} = DM_{gm}(S)$ of Voevodsky's geometric motives (with rational coefficients) over S (see [Voe00]). Besides, $DM_{gm} = \langle Chow \rangle$ (here we consider the full embedding $Chow \to DM_{gm}$ that is a natural extension of the embedding $Chow^{eff} \to DM^{eff}_{gm}$ given by ibid.).*

3. *All $DM_c(X)$ are idempotent complete.*

4. *For any f the following functors are defined: $f^* : DM_c(Y) \leftrightarrows DM_c(X) : f_*$ and $f_! : DM_c(X) \leftrightarrows DM_c(Y) : f^!$; f^* is left adjoint to f_* and $f_!$ is left adjoint to $f^!$.*

   *We call these the **motivic image functors**. Any of them (when f varies) yields a 2-functor from the category of reasonable schemes with separated morphisms of finite type to the 2-category of triangulated categories.*

5. *f^* is symmetric monoidal; $f^*(\mathbb{Q}_Y) = \mathbb{Q}_X$.*

6. *$f_* \cong f_!$ if f is proper.*

   If f is an open immersion, we have $f^! = f^*$.

7. *If X, Y are regular, and $\mathcal{O}_X$ is a free finite-dimensional $\mathcal{O}_Y$-module, then the adjunction morphism $M \to f_*f^*(M)$ splits for any $M \in \operatorname{Obj} DM_c(Y)$.*

8. *If $i : Z \to X$ is a closed embedding, $U = X \setminus Z$, $j : U \to X$ is the complementary open immersion, then the motivic image functors yield gluing data for $DM_c(-)$ (in the sense of Definition 1.5.1(1); one should set $\underline{C} = DM_c(S)$, $\underline{D} = DM_c(Z)$, and $\underline{E} = DM_c(U)$ in it).*

9. *$DM_c(S)$ (as a triangulated category) is generated by $\{g_*(\mathbb{Q}_X)(r)\}$, where $g : X \to S$ runs through all projective morphisms such that X is regular, $r \in \mathbb{Z}$.*

10. *The functor g^* can be defined for any (separated) morphism g not necessarily of finite type; this definition respects the composition for morphisms.*

    Moreover, one can also define $j_K^!$ for $K \in \mathcal{S}$ (see Notation). Besides, if for composable morphisms f, g (not necessarily of finite type) all of $f^!, g^!, (f \circ g)^!$ are defined (i.e. any of $f, g, f \circ g$ is either of finite type or of the type j_K), then $(f \circ g)^! \cong g^! \circ f^!$.

11. *Let S be a scheme which is the limit of an essentially affine projective system of schemes S_β. Then $DM_c(S)$ is isomorphic to the 2-colimit of the categories $DM_c(S_\beta)$; in these isomorphism all the connecting functors are given by the corresponding $(-)^*$ (cf. the previous assertion).*

12. *The family of functors j_K^*, where K runs through $\mathcal{S}$ (see Notation), is conservative on $DM_c(S)$.*

13. *In the setting of assertion 8, for any $M, N \in \operatorname{Obj} DM(S)$ there exists a complex $DM_c(Z)(i^*(M), i^!(N)) \to DM_c(S)(M, N) \to DM_c(U)(j^*(M), j^*(N))$ (of abelian groups) that is exact in the middle.*

14. *For any $l \in \mathbb{P}$ we have an exact functor $\mathcal{H}^{et}_{\mathbb{Q}_l}(S) : DM_c(S) \to D^b Sh^{et}(S[1/l], \mathbb{Q}_l)$. Moreover, the collection of $\mathcal{H}^{et}_{\mathbb{Q}_l}(-)$ converts any of the (four) types of the motivic image functors (when they are defined) into the corresponding (derived) étale sheaf functor (we will never include R into the notation for those).*

Proof. Most of these statements were stated in the introduction of [CiD09] (and proved later in ibid.); see §1.1 of [Bon10c] for more detail.

Assertion 7 was established in process of the proof of Theorem 14.3.3 of [CiD09].

The first part of assertion 2 is given by Corollary 16.1.6 of ibid. The second part of it was proved in §6.4 of [Bon09].

Assertion 12 easily follows from Theorem 2.2.1(IV) below.

Assertion 14 will be proved in detail in a forthcoming paper of Deglise and Cisinski. Alternatively, note that Corollary 16.2.16 of [CiD09] allows to reduce it to the corresponding properties of the version of $\mathcal{H}^{et}_{\mathbb{Q}_l}(-)$ defined on certain categories $D_{\mathbb{A}^1}(-,\mathbb{Q})$ (in the notation of loc.cit.). The latter were verified in [Ayo11] (these matters are thoroughly discussed in §6 of [Kah12]). $\square$

2.2 The Chow weight structure for $DM_c(S)$

We define $Chow(S)$ as the Karoubi-closure of $\{f_*(\mathbb{Q}_X)(r)[2r]\}$ in $DM_c(S)$; here $f : X \to S$ runs through all finite type projective morphisms such that X is regular, $r \in \mathbb{Z}$.

In [Bon10c] the following results were proved; most of them were also independently (and somewhat earlier) established in [Heb11].

Theorem 2.2.1. *I There exists a (unique) bounded weight structure w_{Chow} for $DM_c(S)$ whose heart is $Chow(S)$.*

II Let $f : X \to Y$ be a (separated) finite type morphism of schemes. Then the following statements are valid.

1. $f^!$ and f_* are right weight-exact; f^* and $f_!$ are left weight-exact.

2. Suppose moreover that f is smooth. Then f^* and $f^!$ are also weight-exact.

3. Moreover, f^* is weight-exact for any f that could be presented as a projective limit of smooth morphisms such that the corresponding connecting morphisms are smooth affine.

III Let K be a generic point of S, $M \in Obj DM_c(S)$.

1. Suppose that $j_K^* M \in DM_c(K)_{w_{Chow}\ge 0}$ (resp. $j_K^* \in DM_c(K)_{w_{Chow}\le 0}$). Then there exists an open immersion $j : U \to S$, $K \in U$, such that $j^*M \in DM_c(U)_{w_{Chow}\ge 0}$ (resp. $j^*M \in DM_c(U)_{w_{Chow}\le 0}$).

2. Suppose that $j_K^* M \in DM_c(K)_{w_{Chow}=0}$. Then there exists an open immersion $j : U \to S$, $K \in U$, such that j^*M is a retract of $(g\circ h)_*\mathbb{Q}_P(s)[2s]$, where $h : P \to U'$ is a smooth projective morphism, U' is a regular scheme, $g : U' \to U$ is a finite universal homeomorphism, $s \in \mathbb{Z}$.

IV $M \in DM_c(S)_{w_{Chow}\ge 0}$ (resp. $M \in DM_c(S)_{w_{Chow}\le 0}$) if and only if for any $K \in \mathcal{S}$ we have $j_K^!(M) \in DM_c(K)_{w_{Chow}\ge 0}$ (resp. $j_K^*(M) \in DM_c(K)_{w_{Chow}\le 0}$).

Proof. See Theorems 2.1.2(I) and 2.2.1(II, III), Lemma 2.2.4, Remark 2.3.6(4), and Proposition 2.2.3 of [Bon10a], respectively. For an f that is not quasi-

projective assertion II was proved in (Theorem 3.7 of) [Heb11]; yet we will not actually need non-quasi-projective morphisms below. □

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Remark 2.2.2. Note also that an alternative construction of w_{Chow} over any (not necessarily reasonable) excellent separated finite-dimensional scheme S was considered in §2.3 of [Bon10c]. Its functoriality properties were only studied only with respect to quasi-projective morphisms; yet this is quite sufficient for our purposes.

Below we will call weight spectral sequences and weight filtrations corresponding to w_{Chow} the *Chow-weight* ones.

2.3 On weights for S-sheaves, Chow-weight spectral sequences and their degeneration

Below we will need certain 'weights' for $\mathcal{H}^{et}_{\mathbb{Q}_l}$. First we recall that in certain cases weights are defined on $D^b_cSh^{et}(S,\mathbb{Q}_l)$.

Proposition 2.3.1. *I Let S be a finite type (separated)* $\operatorname{Spec}\mathbb{F}_p$*-scheme (for a prime $p\neq l$).*

Then there exist certain $D^b_cSh^{et}(S,\mathbb{Q}_l)_{w\le 0}$, $D^b_cSh^{et}(S,\mathbb{Q}_l)_{w\ge 0}\subset Obj D^b_cSh^{et}(S,\mathbb{Q}_l)$, that satisfy the following properties.

1. *For any $m\in\mathbb{Z}$ denote $D^b_cSh^{et}(S,\mathbb{Q}_l)_{w\le 0}[m]$ by $D^b_cSh^{et}(S,\mathbb{Q}_l)_{w\le m}$ and denote $D^b_cSh^{et}(S,\mathbb{Q}_l)_{w\ge 0}[m]$ by $D^b_cSh^{et}(S,\mathbb{Q}_l)_{w\ge m}$.*

   Then for $X\in Obj D^b_cSh^{et}(S,\mathbb{Q}_l)$ we have: $X\in D^b_cSh^{et}(S,\mathbb{Q}_l)_{w\le m}$ (resp. $X\in D^b_cSh^{et}(S,\mathbb{Q}_l)_{w\ge m}$) whenever for any $j\in\mathbb{Z}$ we have $H^t_j(X)\in D^b_cSh^{et}(S,\mathbb{Q}_l)_{w\le m+j}$ (resp. $H^t_j(X)\in D^b_cSh^{et}(S,\mathbb{Q}_l)_{w\ge m+j}$).

2. *Denote $D^b_cSh^{et}(S,\mathbb{Q}_l)_{w\le m}\cap D^b_cSh^{et}(S,\mathbb{Q}_l)^{t=0}$ by $Sh^{et}_{per}(S,\mathbb{Q}_l)_{w\le m}$, and denote $D^b_cSh^{et}(S,\mathbb{Q}_l)_{w\ge m}\cap D^b_cSh^{et}(S,\mathbb{Q}_l)^{t=0}$ by $Sh^{et}_{per}(S,\mathbb{Q}_l)_{w\ge m}$; $Sh^{et}_{per}(S,\mathbb{Q}_l)_{w=m}=Sh^{et}_{per}(S,\mathbb{Q}_l)_{w\le m}\cap Sh^{et}_{per}(S,\mathbb{Q}_l)_{w\ge m}$. Then $Sh^{et}_{per}(S,\mathbb{Q}_l)_{w=m}$ yield exact abelian subcategories of $Sh^{et}_{per}(S,\mathbb{Q}_l)$, that contain all $Sh^{et}_{per}(S,\mathbb{Q}_l)$-subquotients of their objects. Besides, for $j\neq m\in\mathbb{Z}$ we have $Sh^{et}_{per}(S,\mathbb{Q}_l)_{w=m}\perp Sh^{et}_{per}(S,\mathbb{Q}_l)_{w=j}$.*

3. *For any open immersion $j:U\to S$ and $m\in\mathbb{Z}$ (the perverse sheaf version of) the functor $j_{!*}$ sends $Sh^{et}_{per}(S,\mathbb{Q}_l)(U)_{w\le m}$ into $Sh^{et}_{per}(S,\mathbb{Q}_l)_{w\le m}$, and sends $Sh^{et}_{per}(S,\mathbb{Q}_l)(U)_{w\ge m}$ into $Sh^{et}_{per}(S,\mathbb{Q}_l)_{w\ge m}$.*

4. *$\mathcal{H}^{et}_{\mathbb{Q}_l}$ is 'weight-exact' i.e. it sends $DM_c(S)_{w_{Chow}\le m}$ into $D^b_cSh^{et}(S,\mathbb{Q}_l)_{w\le m}$, and sends $DM_c(S)_{w_{Chow}\ge m}$ into $D^b_cSh^{et}(S,\mathbb{Q}_l)_{w\ge m}$.*

II Let S be a finite type (separated) $\operatorname{Spec}\mathbb{Q}$*-scheme. Present S as a inverse limit of finite type* $\operatorname{Spec}\mathbb{Z}[\frac{1}{l}]$*-schemes S_i, and define $\tilde{\mathcal{H}}^{et}_{\mathbb{Q}_l}$ as the direct limit of $\mathcal{H}^{et}_{\mathbb{Q}_l S_i}$ (its target $\tilde{D}^b_c Sh^{et}(S,\mathbb{Q}_l)$ is the 2-colimit of the corresponding $D^b_c Sh^{et}(S_i,\mathbb{Q}_l)$). Then $\mathcal{H}^{et}_{\mathbb{Q}_l}$ can be factorized through $\tilde{\mathcal{H}}^{et}_{\mathbb{Q}_l}$. Moreover, $\tilde{D}^b_c Sh^{et}(S,\mathbb{Q}_l)$ possesses a (perverse) t-structure that is compatible with t (with respect to this connecting functor). Lastly, for $\tilde{D}^b_c Sh^{et}(S,\mathbb{Q}_l)$ one can define weights such that the analogues of all of the assertions of part 1 are fulfilled.*

Proof. I All the assertions expect (4) are well-known properties of weights of mixed sheaves that were established in §5 of [BBD82], whereas assertion 4 was verified in §3.6 of [Bon10c].

II The t-exactness of the connection functor $\tilde{D}^b_c Sh^{et}(S,\mathbb{Q}_l)\to D^b_c Sh^{et}(S,\mathbb{Q}_l)$ easily follows from the standard properties of the perverse t-structure. Everything else was verified in §3 of [Hub97], expect assertion 4 that was established in §3.4 of [Bon10c] (in this case).

□

Now we prove the main properties of Chow-weight spectral sequences for $H^{et}_{\mathbb{Q}_l,0}$. To this end we state the following conjecture.

Conjecture 2.3.2. *The spectral sequence $T_{w_{Chow}}(\mathcal{H}^{et}_{\mathbb{Q}_l}, M)$ degenerates at E_2 for any $M\in Obj DM_c(S)$ (for any reasonable S).*

Theorem 2.3.3. *I1. Let S be a finite type separated scheme over* $\operatorname{Spec}\mathbb{F}_p$*, $M\in Obj DM_c(S)$, $H=H^{et}_{\mathbb{Q}_l,0}$. Then $E_s^{pq}T_{w_{Chow}}(H,M)\in Sh^{et}_{per}(S,\mathbb{Q}_l)_{w=q}$ for any $p,q\in\mathbb{Z}$, $s>0$. Besides, $(W_m H)(M)$ is a filtration of $H(M)$ whose m-th factor belongs to $Sh^{et}_{per}(S,\mathbb{Q}_l)_{w=m}$ (for all $m\in\mathbb{Z}$). Moreover, this filtration is uniquely and functorially characterized by the latter property.*

2. For S being separated over $\operatorname{Spec}\mathbb{Q}$ *and $H=\tilde{H}^{et}_{\mathbb{Q}_l,0}$ the (obvious) analogue of assertion I1 is fulfilled.*

II Let S be a characteristic p (reasonable) scheme ($p\neq l$; it could be 0).

Then the following statements are valid.

1. Conjecture 2.3.2 holds.

2. Let $j:U\to S$ be an open embedding; denote the complimentary closed embedding by i. For $M\in DM_c(U)_{w_{Chow}\ge s}$ (resp. $M\in DM_c(U)_{w_{Chow}\le s}$) suppose that $H^{et}_{\mathbb{Q}_l,m}(M)=0$ for all $m\neq 0$.

Then $(W_{s+m+1}H^{et}_{\mathbb{Q}_l,m})(i^! j_! M)=0$ for any $m>0$ (resp. $(W_{s+m}H^{et}_{\mathbb{Q}_l,m})(i^* j_* M)=H^{et}_{\mathbb{Q}_l,m}(i^* j_* M)$ for any $m<0$).

Proof. I1. Proposition 2.3.1(4) yields that $E_1^{pq}(T)\in Sh^{et}_{per}(S,\mathbb{Q}_l)_{w=q}$.

Hence the same is true for $E_s^{pq}(T)$ for any $s \geq 1$ (since E_s^{pq} is a subfactor of $E_1^{pq}(T)$; here we apply Proposition 2.3.1(2)).

Hence $E_\infty^{p+q}(T) \in Sh^{et}_{per}(S, \mathbb{Q}_l)_{w=q}$ also, and we obtain that the factors of the Chow-weight filtration are of the weights prescribed. Now, the orthogonality of (subquotients of) perverse sheaves of distinct weight yields that this condition determines the filtration in a functorial way.

The same argument proves assertion I2.

II1. We verify the degeneration of $T_{w_{Chow}}(M, \mathcal{H}^{et}_{\mathbb{Q}_l})$ for some fixed $M \in Obj DM_c(S)$. We note that t-exact conservative functors cannot kill non-zero morphisms in the heart (of the corresponding) t-structure, whereas for any affine Zariski (or étale) cover $f : S' \to S$ the functor $f^* : D^b_c Sh^{et}(S, \mathbb{Q}_l) \to D^b_c Sh^{et}(S_0, \mathbb{Q}_l)$ is conservative and t-exact with respect to the corresponding perverse t-structures. Hence it suffices to prove the statement with S replaced by S'. Here we apply Proposition 2.1.1(14), Theorem 2.2.1(II2), and Proposition 1.3.2(I) in order to prove that $f^* T_{w_{Chow}}(M, H^{et}_{\mathbb{Q}_l,0}) \cong T_{w_{Chow}}(f^*M, H^{et}_{\mathbb{Q}_l,0})$. Therefore, we may assume S to be affine. Moreover, we can assume it to be a (filtered) projective limit of schemes that are finitely generated over the corresponding prime field (i.e. over $\operatorname{Spec} \mathbb{F}_p$ or $\operatorname{Spec} \mathbb{Q}$) such that the connecting morphisms are smooth and affine. Indeed, we can assume that the corresponding S_0 (see the definition of reasonability for schemes) is affine also. Then the celebrated theorem of Popescu (see Theorem 1.8 of [Pop86]) easily implies that $S_0 = \varprojlim S_0^i$ for some (regular) S_0^i that are finitely generated over $\operatorname{Spec} \mathbb{F}_p$ or over $\operatorname{Spec} \mathbb{Q}$, where the connecting morphisms are smooth affine. Here we use the fact that the morphism $S_0 \to \operatorname{Spec} \mathbb{F}_p$ or $S_0 \to \operatorname{Spec} \mathbb{Q}$, respectively, is obviously *regular* (in the sense of ibid.).

Then we can assume that S is defined over all S_0^i; the connecting morphisms will be smooth affine also. Note that all the corresponding S^i/S_0^i are reasonable.

Hence we can assume that S_0 (and so, S) is of finite type over $\operatorname{Spec} \mathbb{F}_p$ (for $p \neq l$) or over $\operatorname{Spec} \mathbb{Q}$. Here we apply Proposition 2.1.1(11) in order to find an index i such that $M = M'_{S^i}$ for some $M' \in Obj DM_c(S^i)$. Note here that S is pro-smooth over S^i; hence there exists an isomorphism of spectral sequences similar to the one described in the previous reduction step above (we only have to shift the homology by $\dim S_0 - \dim S_0^i$).

Now let S be of finite type over $\operatorname{Spec} \mathbb{F}_p$. In this case assertion I1 yields that $E_s^{pq}(T) \in Sh^{et}_{per}(S, \mathbb{Q}_l)_{w=q}$ (for any $p, q \in \mathbb{Z}$, $s > 0$). Now (by Proposition 2.3.1(2)) there are no non-zero morphisms between distinct $Sh^{et}_{per}(S, \mathbb{Q}_l)_{w=m}$. Hence all the connecting morphisms for $E_s(T)$ vanish for all $s > 1$, and we obtain the result.

In the case when S is of finite type over $\operatorname{Spec} \mathbb{Q}$ we note that the same argument proves the degeneration of $T(\tilde{H}^{et}_{\mathbb{Q}_l,0}, -)$; hence the functoriality of

Chow-weight spectral sequences (with respect to H) yields the assertion desired.

2. The same reduction arguments as above enable us to assume that S is of finite type over $\operatorname{Spec}\mathbb{F}_p$ (for $p \neq 0$) or over $\operatorname{Spec}\mathbb{Q}$. In this case Proposition 2.1.1(14) along with assertion I allow us to translate our assertion into the corresponding analogue for weights on $D^b_c Sh^{et}(S, \mathbb{Q}_l)$. Now, for $S/\operatorname{Spec}\mathbb{F}_p$ we have $i^! D^b_c Sh^{et}(S, \mathbb{Q}_l)_{w\geq s} \subset D^b_c Sh^{et}(Z, \mathbb{Q}_l)_{w\leq s}$ and $i^* D^b_c Sh^{et}(S, \mathbb{Q}_l)_{w\geq s} \subset D^b_c Sh^{et}(Z, \mathbb{Q}_l)_{w\geq s}$ by (§5.1.14, (i*) and (i), of) [BBD82]. Proposition 1.5.3(IV2) yields: it suffices to note that $j_{!*}$ respects weights of mixed sheaves; this is Corollary 5.3.2 of [BBD82].

For $S/\operatorname{Spec}\mathbb{Q}$ it suffices to verify the assertion for $\tilde{H}^{et}_{\mathbb{Q}_l,m}$ instead of $H^{et}_{\mathbb{Q}_l,m}$. In this setting we can apply the Remark succeeding Definition 3.3 and Corollary 3.5 of [Hub97] (instead of the results of [BBD82] cited above).

□

Remark 2.3.4. 1. Using étale descent, one could reduce (the general case) of the conjecture the case when S is a mixed characteristic local (or even henselian) scheme. The author suspects that in this latter situation the conjecture is closely related with Deligne's weight-monodromy conjecture.

2. Using Verdier duality (for motives or sheaves) one can easily carry over the results above from étale homology to étale cohomology.

3. In the characteristic 0 case of Theorem 2.3.3(II), we could have tried to use M. Saito's Hodge modules in our weight arguments (in order to avoid the usage of $\tilde{\mathcal{H}}^{et}_{\mathbb{Q}_l}$). The main problem here is that (to the knowledge of the author) no 'Hodge module realization' of motives is known to exist at the moment (still see the proof of Proposition 7.6 of [Wil12] for a certain reasoning avoiding this difficulty).

Alternatively, one could try to reduce the characteristic 0 case to the positive characteristic one using the methods and results of §6 of [BBD82].

3 On the existence of a (nice) motivic t-structure

In §3.1 we define a (motivic) t-structure t_l for $DM_c(S)$ as the one that is strictly compatible with the perverse t-structure for the $\mathbb{Q}_l$-étale homology (cf. §2.10 of [Bei98]). We also study the functoriality of this definition.

In §3.2 we reduce the existence of t_l to the case when S is the (spectrum of) a universal domain (of characteristic distinct from l). Moreover, the existence of t_l over universal domains automatically yields that Chow-weight filtrations and Chow-weight spectral sequences can be lifted from $Sh^{et}_{per}(S, \mathbb{Q}_l)$ to motives. When S is an equicharacteristic scheme, the weight filtration for $\underline{Ht}_l$ obtained this way is strictly compatible with morphisms.

In §3.3 we study certain properties of motives that follow from the *niceness* of t_l (i.e. from its transversality with w_{Chow}).

In §3.4 we apply these results (in a certain Noetherian induction step). We prove that a nice t_l exists over an arbitrary reasonable scheme S of characteristic p if such a t_l exists over some universal domain of the same characteristic.

3.1 The motivic t-structure (for $S/\operatorname{Spec}\mathbb{Z}[\frac{1}{l}]$)

Till §4.3 we will fix some prime l, and will usually assume that all the schemes we consider are $\operatorname{Spec}\mathbb{Z}[\frac{1}{l}]$-ones. In this case we will define the motivic t-structure in terms of $\mathcal{H}^{et}_{\mathbb{Q}_l}$; we will treat the question whether it actually depends on l later.

Definition 3.1.1. Let S be a (reasonable) scheme. We consider the category $D^b_cSh^{et}(S[1/l],\mathbb{Q}_l)$ (using the results of §6 of [Eke90]), and denote by t the perverse t-structure (with respect to the middle perversity) for it. We define t combining the results of ibid. with those of [Gab04].

1. Consider the class $DM_c(S)^{t_l\le 0}$ (resp. $DM_c(S)^{t_l\ge 0}$) consisting of those $M\in ObjDM_c(S)$ that satisfy: $\mathcal{H}^{et}_{\mathbb{Q}_l}(M)\in D^b_cSh^{et}(S[1/l],\mathbb{Q}_l)^{t\le 0}$ (resp. $\mathcal{H}^{et}_{\mathbb{Q}_l}(M)\in D^b_cSh^{et}(S[1/l],\mathbb{Q}_l)^{t\ge 0}$).

2. For a $\operatorname{Spec}\mathbb{Z}[\frac{1}{l}]$-scheme S if $(DM_c(S)^{t_l\le 0},\ DM_c(S)^{t_l\ge 0})$ yield a t-structure for $DM_c(S)$, we will say that (the t-structure) t_l exists for $DM_c(S)$, or that it exists over S. We will denote the heart of t_l (in this case) by $MM(S)$.

3. We will use the term "(left, right, or both) t-exact functor" for functors between certain $DM_c(-)$ that respect (the 'halves of') t_l in the corresponding way without (necessarily) assuming that t_l yields a t-structure.

4. If t_l exists for $DM_c(S)$, we will say that it is *nice* if it is transversal to w_{Chow}.

Remark 3.1.2. 1. If t_l exists over S, then it is automatically bounded, since the étale homology of any object of $DM_c(S)$ is. The latter fact is immediate from Proposition 2.1.1((9), (14)).

In particular, we obtain that t_l is non-degenerate.

2. We restrict ourselves to reasonable schemes for two reasons.

Firstly, we need a nice formalism of constructible $\mathbb{Q}_l$-sheaves; here we follow §6 of [Eke90]. Yet it seems that recent results of O. Gabber (unpublished; still see [Ill07]) allow to lift the condition of the existence of S_0.

Secondly, we used the existence of some S_0 in the proof of Theorem 2.3.3(II). Yet it could be higher-dimensional; in this more general case one should use the construction of w_{Chow} described in §2.3 of [Bon10c] (since

in §2.2 of ibid. and in [Heb11] S was required to be reasonable, though in a sense that is somewhat more general than the one we use in the current paper; cf. Remark 2.2.2).

We will need certain functoriality properties of $(DM_c(-)^{t_l\le 0},\ DM_c(-)^{t_l\ge 0})$ below; certainly, they become even more interesting (for themselves) if t_l exists.

Lemma 3.1.3. *Let $f : X \to Y$ be a morphism of schemes. Then the following statements are valid.*

1. If f is an immersion, then f_* and $f^!$ are left t-exact, whereas $f_!$ and f^* are right t-exact (see Definition 3.1.1(3)).

2. If f is affine, then $f_!$ is left t-exact, and f_* is right t-exact.

3. If f is quasi-finite affine, then f_* and $f_!$ are t-exact.

4. If f is proper of relative dimension $\le d$, then $f_*[d](=f_![d])$ is left t-exact, and $f_*[-d]$ is right t-exact.

5. If f is smooth of dimension d, then $f^![-d]$ and $f^*[d]$ are t-exact.

6. If K is a point of S of dimension d (see the Notation), then $j_K^*[-d]$ (resp. $j_K^![-d]$) is left (resp. right) t-exact.

7. Moreover, for $M \in Obj DM_c(S)$ we have $M \in DM_c(S)^{t_l\le 0}$ (resp. $M \in DM_c(S)^{t_l\ge 0}$) whenever for any $K \in \mathcal{S}$, K is of dimension d, we have $j_K^* M[-d] \in DM_c(K)^{t_l\le 0}$ (resp. $j_K^! M[-d] \in DM_c(K)^{t_l\ge 0}$).

8. For a closed embedding $i : Z \to S$ and the complimentary immersion $j : U \to S$ for $M \in Obj DM_c(S)$ we have: $M \in DM_c(S)^{t_l\le 0}$ (resp. $M \in DM_c(S)^{t_l\ge 0}$) whenever $j^*M \in DM_c(U)^{t_l\le 0}$ and $i^*M \in DM_c(Z)^{t_l\le 0}$ (resp. $j^*M \in DM_c(U)^{t_l\ge 0}$ and $i^!M \in DM_c(Z)^{t_l\ge 0}$).

Proof. Proposition 2.1.1(14, 6) reduces assertions 1–7 to the corresponding properties of the perverse t-structure. The latter follow from its well-known functoriality properties. Note that their proofs in [BBD82] can be carried over to our more general situation without difficulty. To this end we mainly need the definition of t, the properties of the 'canonical' t-structure for $D_c^bSh^{et}(S[1/l],\mathbb{Q}_l)$ (see Theorem 6.3(i) of [Eke90]) and Verdier duality. Analogues of the results needed to carry over the proof from [BBD82] to our context were (mostly) proved by O. Gabber; see [Ill07].

(8): Proposition 2.1.1(14) yields: it suffices to verify that the perverse t-structure for $D_c^bSh^{et}(S[1/l],\mathbb{Q}_l)$ can be glued from those for $D_c^bSh^{et}(U[1/l],\mathbb{Q}_l)$ and $D_c^bSh^{et}(Z[1/l],\mathbb{Q}_l)$ (see Definition 1.5.1(4)). Now, Theorem 6.3 of [Eke90] yields that the these categories along with the corresponding connecting functors yield a gluing data. By Proposition 1.5.3(II2) it remains to verify that $i_{*,D_c^bSh^{et}(-,\mathbb{Q}_l)}$ and $j^*_{D_c^bSh^{et}(-,\mathbb{Q}_l)}$ are t-exact. Now, these properties of the perverse t-structure are well-known (cf. assertion 1 and Remark 6.1(2) of [Gab04]).

□

Remark 3.1.4. Actually, below we will not use all of the Lemma. We will not use assertions 2, 3, and 8, and we only apply assertion 4 when f is a finite morphism (of spectra of fields). So, instead of the results [Ill07] we could have applied some partial cases of the corresponding statements, that are (more) well-known.

Now we formulate the first of the main results of this paper.

Theorem 3.1.5. *Suppose that for any point K of (a reasonable* $\operatorname{Spec}\mathbb{Z}[\frac{1}{l}]$*-) scheme S there exists t_l for $DM_c(K)$.*

Then t_l exists for $DM_c(S)$ also.

3.2 The proof of the 'globalization' theorem for t_l

Till §4.3 we will assume that S is a (reasonable) $\operatorname{Spec}\mathbb{Z}[\frac{1}{l}]$-scheme.

We will need the following statement.

Lemma 3.2.1. *Let K be a generic point of a scheme U' whose dimension is d (see Notation); denote the morphism $K \to U'$ by j_K.*

Let $M \in Obj DM_c(U')$, and suppose that the $j_K^*M[-d] \in DM_c(K)^{t_l\le 0}$ (resp. $j_K^*M[-d] \in DM_c(K)^{t_l\ge 0}$). Then there exists an open immersion $j: U'' \to U'$, $K \in U''$, such that $j^*M \in DM_c(U'')^{t_l\le 0}$ (resp. $j^*M \in DM_c(U'')^{t_l\ge 0}$).

Proof. Part 14 of ibid. reduces this fact to its $D^b_cSh^{et}(S,\mathbb{Q}_l)$-version. Applying the Verdier duality, we obtain that it suffices to verify the following statement: for any $C \in Obj D^b_cSh^{et}(U',\mathbb{Q}_l)$ if $j_K^*(C)[-d] \in D^b_cSh^{et}(K,\mathbb{Q}_l)^{t\le 0}$, then there exists an open immersion $j: U'' \to U'$, $K \in U'$, such that $j^*C \in D^b_cSh^{et}(U'',\mathbb{Q}_l)^{t\le 0}$. Now, over K the perverse t-structure for $D^b_cSh^{et}(K,\mathbb{Q}_l)$ coincides with the 'canonical' one (corresponding to the canonical t-structure for the derived category $D^b_cSh^{et}(K,\mathbb{Q}_l)$), whereas over any U'' any ('ordinary') constructible $\mathbb{Q}_l$-sheaf belongs to $D^b_cSh^{et}(U'',\mathbb{Q}_l)^{t\le 0}$. Considering the canonical homology of C (note that j^* and j_K^* are exact when restricted to the category of 'ordinary' $\mathbb{Q}_l$-sheaves) we obtain that it suffices to verify: if the stalk of some constructible $\mathbb{Q}_l$-sheaf T at K is zero, then for some open $U'' \subset U'$, $K \in U''$, we have $j^*T = 0$. This is immediate from Proposition I.12.10 of [FrK88]. Note here: one can apply the method of the proof of loc. cit. in our (more general) case by Theorem 6.3(i) of [Eke90]; cf. also Remark 9.5 of [Gab04].

□

Now we prove Theorem 3.1.5.

We should prove that $(DM_c(S)^{t_l\le 0}, DM_c(S)^{t_l\ge 0})$ (see Definition 3.1.1(1)) yield a t-structure for $DM_c(S)$.

Obviously, to this end it suffices to verify that for $DM_c(S)^{t_l\le 0}$ and $DM_c(S)^{t_l\ge 0}$ prescribed by Definition 3.1.1 we have the orthogonality property, and that t_l-decompositions exist.

The proof of orthogonality uses an argument contained in the proof of Proposition 2.2.3 of [Bon10c]. We apply noetherian induction. Suppose that the assertion is fulfilled over any (proper) closed subscheme of S.

For any (fixed) $M \in DM_c(S)^{t_l\le 0}$, $N \in DM_c(S)^{t_l\ge 1}$, $h \in DM_c(S)(M,N)$, we should prove that $h = 0$.

Let K be a generic point of S of dimension d. Lemma 3.1.3(6) yields that $j_K^* M[-d] \in DM_c(K)^{t_l\le 0}$, $j_K^! N[-d] \in DM_c(K)^{t_l\ge 1}$. Hence $j_K^* h = 0$ (since t_l exists for K-motives). Hence (by Proposition 2.1.1(11)) there exists an open immersion $j : U \to S$, $K \in U$, such that $j(h) = 0$. Let $i : Z \to S$ denote the complimentary closed embedding; Lemma 3.1.3(1) yields that $i^*(M) \in DM_c(Z)^{t_l\le 0}$, $i^! N \in DM_c(Z)^{t_l\ge 1}$. By the inductive assumption (applied to Z) we have $DM_c(Z)(i^*(M), i^!(N)) = 0$. Hence Proposition 2.1.1(13) yields the assertion.

It remains to verify the existence of a t_l-decomposition for an $M \in Obj DM_c(S)$. We use the method of the proof similar to that of Proposition 2.3.3 of [Bon10c]. Again, we apply noetherian induction and assume that the assertion is fulfilled over any proper closed subscheme of S.

We choose some generic point K of S. We consider the t_l-decomposition

$$A_K[-d] \to j_K^* M[-d] \to B_K[-d] \tag{5}$$

(of $j_K^* M[-d]$ in $DM_c(K)$). We verify that there exists an open immersion $j : U \to S$ containing S such that (5) (shifted by $[d]$) lifts to a t_l-decomposition of $j^* M$. By Proposition 2.1.1(11) it suffices to verify: for any open $U' \subset S$ containing K and any $A_{U'}, B_{U'} \in DM_c(U')$ such that the 'restriction' of $(A_{U'}, B_{U'})$ to K equals (A_K, B_K), there exists an open $U \subset U'$ (containing K) such that 'restrictions' A_U, B_U of $A_{U'}, B_{U'}$ to U belong to $DM_c(U)^{\le 0}$ and to $DM_c(U)^{\ge 1}$, respectively. This is immediate from Lemma 3.2.1.

Again, we consider the closed embedding $i : Z \to S$ complimentary to j. Now, the idea is that t_l for $DM_c(S)$ can be glued from those for $DM_c(U)$ and $DM_c(Z)$. Though we only have t_l-decompositions in the latter category (by the inductive assumption), this is sufficient to construct the t_l-decomposition of M. Indeed, by Proposition 1.5.3(I) there exists a distinguished triangle $A \to M \to B$ such that $j^* A \in DM_c(U)^{t_l\le 0}$ and $i^* A \in DM_c(Z)^{t_l\le 0}$ (resp. $j^* B \in DM_c(U)^{t_l\ge 0}$ and $i^! B \in DM_c(Z)^{t_l\ge 0}$). By Lemma 3.1.3(7), this triangle yields the t_l-decomposition of M.

Remark 3.2.2. 1. Actually, we do not need a complete characterization of t_l for the proof. We only need a pointwise characterization of t_l (cf. Lemma 3.1.3(7)) and Lemma 3.2.1 for it.

Also note here: if we have **any** t-structures for $DM_c(S)$, $DM_c(K)$, and $DM_c(U)$ for any U such that all possible j^* and $j_K^*[-d]$ are t-exact, then the statement of Lemma 3.2.1 for these t-structures is fulfilled automatically. Indeed, Proposition 2.1.1(11) implies that $j^*(M^{\tau\geq 1})$ (resp. $j^*(M^{\tau\leq -1})$) vanishes for some U, since $j_K^*(M^{\tau\geq 1})$ (resp. $j_K^*(M^{\tau\leq -1})$) does.

Still, the author does not know how to verify Lemma 3.2.1 for the version of (the description of) the motivic t-structure (over fields) given by Proposition 4.5 of [Bei98].

2. Lemma 3.1.3(7) yields that t_l does not depend on the choice of (a version of) $\mathcal{H}^{et}_{\mathbb{Q}_l}$ over S; see also Remark 4.1.2 below.

Now we prove that it suffices to verify the conservativity of $\mathcal{H}^{et}_{\mathbb{Q}_l}$ and the existence of t_l over universal domains.

Proposition 3.2.3. *1. Let $\mathcal{H}^{et}_{\mathbb{Q}_l}$ be conservative on $DM_c(K)$ for all $K \in \mathcal{S}$. Then the same is true for $DM_c(S)$.*

2. Suppose that $\mathcal{H}^{et}_{\mathbb{Q}_l}$ is conservative over some set of universal domains K_i of certain characteristics $p_i \neq l$ (p_i could be 0).

Then $\mathcal{H}^{et}_{\mathbb{Q}_l}$ is conservative over any (reasonable) S that satisfies the following conditions: the characteristic of any point of S is one of p_i.

3. Suppose that t_l exist over some universal domains K_i of characteristics p_i.

Then t_l also exists over any (reasonable) S as in the previous assertion.

Proof. 1. Immediate from Proposition 2.1.1(12,14).

2. The previous assertion yields: it suffices to verify that $\mathcal{H}^{et}_{\mathbb{Q}_l}$ is conservative over any characteristic p field K (here p could be 0) if it exists over some universal domain K' of characteristics p. We prove the latter statement using a certain 'spreading out' argument.

We should verify: if $\mathcal{H}^{et}_{\mathbb{Q}_l}(M) = 0$ for some $M \in Obj DM_c(K)$, then $M = 0$. We fix some M.

First we note that any object (and morphism) in $DM_c(K)$ is defined over some finitely generated subfield F of K; this is easy from Proposition 2.1.1(11,9). Besides, for any extension of fields the corresponding base change functor for sheaves is conservative (immediate from SGA 4, Exposé VIII, Proposition 9.1). Hence we obtain that $\mathcal{H}^{et}_{\mathbb{Q}_l}(M_F) = 0$ for the corresponding $M_F \in DM_c(\operatorname{Spec} F)$. Therefore, we may assume that $K \subset K'$.

We denote the morphism $\operatorname{Spec} K' \to \operatorname{Spec} K$ by b. Applying the conservativity of base change for sheaves again, we obtain: it suffices to check that b^* is conservative. This is immediate from Lemma 3.4.3 below.

3. Theorem 3.1.5 implies: it suffices to verify that t_l exists over any characteristic p field K (here p could be 0) if it exists over some universal domain K' of characteristics p. We prove this using an argument that is rather similar to the one above.

Again, in order to prove the existence of t_l it suffices to verify the orthogonality axiom and the existence of t_l-decompositions for the classes described in Definition 3.1.1(3).

Arguing as above, we obtain that it suffices to verify: if t_l exists over K', it also exists over its subfield K.

First we consider an algebraically closed $K \subset K'$. Our arguments along with Lemma 3.2.1 yield: if a t_l-decomposition of $Z_{K'}$ for a $Z \in Obj DM_c(K)$ exists in K', a t_l-decomposition of $Z_U[d]$ exists over some smooth connected K-variety U of dimension d (i.e. a t_l-decomposition $Z_1 \to Z_U[-d] \to Z_2$ of $Z_U[-d] = u^*Z[-d]$ exists in $DM_c(U)$, for $u : U \to \operatorname{Spec} K$ being the structure morphism of U). Similarly, if we have a non-zero $h \in DM_c(K)(M,N)$, $M \in DM_c(K)^{t_l \le 0}$, $N \in DM_c(K)^{t_l \ge 1}$, then it vanishes over a certain U (since it vanishes over K').

We denote by $s : \operatorname{Spec} K \to U$ the immersion of some K-point of U to U. Then $h = s^*u^*h$; hence $h = 0$. Now, Lemma 3.1.3(5) yields that $u^*[d]$ is t-exact. Since it is also conservative, we obtain: it suffices to verify that $Z_i = u^*s^*Z_i$ (for $i = 1, 2$). Since $\mathcal{H}^{et}_{\mathbb{Q}_l}$ is conservative by the previous assertion, it suffices to verify that $\mathcal{H}^{et}_{\mathbb{Q}_l}(Z_i) = (u \circ s)^* \mathcal{H}^{et}_{\mathbb{Q}_l}(Z_i)$. Now, it remains to note that $\mathcal{H}^{et}_{\mathbb{Q}_l}(Z_i)$ can be obtained by applying $u^*[d]$ to the t_l-decomposition of $\mathcal{H}^{et}_{\mathbb{Q}_l}(Z)$.

It remains to prove that t_l exists over K if it exists over its algebraic closure. Similarly to the reasoning above, we obtain: for any $Z \in Obj DM_c(K)$ there exists a finite extension F/K (of some degree $d > 0$) such that a t_l-decomposition of Z_F exists (in $DM_c(F)$), and also any $h \in DM_c(K)(M,N)$, $M \in DM_c(K)^{t_l \le 0}$, $N \in DM_c(K)^{t_l \ge 1}$, vanishes over a certain F. Then proposition 2.1.1(7) yields the existence of t_l over K. Indeed, if $f : F \to K$ is the corresponding morphism, loc. cit. yields that the morphism $DM_c(K)(M,N) \to DM_c(F)(f^*M, f^*N)$ is injective. Besides, since f_* is t-exact (see Lemma 3.1.3(4)), we obtain that a t_l-decomposition exists for f_*f^*Z; hence it exists for Z also (since $DM_c(S)^{t_l \le 0}$ and $DM_c(S)^{t_l \ge 0}$ are idempotent complete).

□

We also recall here Proposition 2.1.1(2); it yields that $DM_c(K_i)$ is the category $DM_{gm}(K_i)$ of Voevodsky's motives; hence it suffices to verify the conservativity of $\mathcal{H}^{et}_{\mathbb{Q}_l}$ and the existence of t_l for the latter categories (cf. Remark 4.1.2).

Corollary 3.2.4. *1. Suppose that t_l exist over some universal domains K_i of certain characteristics p_i. Then for any S as in Proposition 3.2.3 Chow-weight filtrations and spectral sequences for $H^{et}_{\mathbb{Q}_l,0}$ over S can be lifted to $MM(S)$.*

2. Suppose that $\operatorname{char} S = p$ *(p is either a prime or 0), and t_l exists for $DM_c(K)$ where K is some universal domain of characteristic p.*

Then there exists a weight filtration for $MM(S)$ with the corresponding functors $W_{\le i,MM}$ such that for any $i \in \mathbb{Z}$ we have: $W_i H^{et}_{\mathbb{Q}_l,0} \cong \mathcal{H}^{et}_{\mathbb{Q}_l} \circ W_{\le i,MM} \circ H_0^{t_l}$.

Proof. 1. Immediate from the functoriality of weight spectral sequences and weight filtrations with respect to exact functors of target categories (see Proposition 1.3.2(I)).

2. By Theorem 2.3.3(II1), Chow-weight spectral sequences degenerate (at E_2) for $H' = H^{et}_{\mathbb{Q}_l,0}$. Hence Proposition 1.3.5(II) yields the degeneration of Chow-spectral sequences also for $H = H_0^{t_l}$. Therefore part I of loc. cit. yields the existence of a weight filtration for $MM = \underline{Ht}_l$ such that $W_i H_0^{t_l} \cong W_{\le i,MM} \circ H_0^{t_l}$. It remains to apply Proposition 1.3.2(I) again.

□

Remark 3.2.5. 1. Certainly, the assumptions of the Corollary also yield that for any open embedding $j : U \to S$ one can lift $j_{!*}$ from $Sh^{et}_{per}(-, \mathbb{Q}_l)$ to $MM(-)$. In particular, if U is regular and dense in S, $j_{!*}\mathbb{Q}_U \in DM_c(S)^{t_l=0}$ could be called the 'intersection motif' of S; it corresponds to the $\mathbb{Q}_l$-adic étale intersection cohomology of S.

2. Conversely to part 2 of the Corollary, suppose that for some (mixed characteristic) S there exists some weight filtration for $MM(S)$ such that $H_i^{t_l}(Chow(S))$ is of weight i (for any $i \in \mathbb{Z}$; this assumption can be reduced to the following ones: $\mathbb{Q}(j)$ is of weight $-2j$ for any $j \in \mathbb{Z}$, whereas $p_!$ respects weights in the corresponding sense for p being a projective morphism of schemes). Then for $H = H_0^{t_l}$ one can easily see that $T(H, -)$ degenerates at E_2 (cf. the proof of Theorem 2.3.3(II1)). Certainly, this yields Conjecture 2.3.2 in this case (see Proposition 1.3.5(II1)). We obtain a good reason to believe Conjecture 2.3.2 (for a general S).

3. Instead of assuming that t_l exists over a universal domain K (of characteristic p), it suffices to assume that it exists over all members of a family K_i of fields such that any finitely generated L of characteristic p embeds into one of K_i. In particular, one could take K_i being algebraically closed fields of characteristics p such that their transcendence degree is not bounded (by any natural number).

3.3 Certain consequences of the existence of a nice motivic t-structure

Now we derive certain consequences from the existence of a nice motivic t-structure for $DM_c(S)$; we will need some of them below in order to make a certain inductive step.

Proposition 3.3.1. *Let a nice t_l exist over S, $m \in \mathbb{Z}$, $M \in ObjDM_c(S)$. Then the following statements are fulfilled.*

I1. The category $MM_m(S) = MM \cap \underline{Hw}_{Chow}[m]$ is (abelian) semi-simple.

2. If $M \in DM_c(S)^{t=0}$, then it possesses an increasing filtration $W_{\le r}M$, $r \in \mathbb{Z}$, whose m-th factor belongs to $MM_m(S)$; this filtration is $\underline{Ht}$-functorial in M.

3. $M \in DM_c(S)_{w_{Chow} \le m}$ (resp. $M \in DM_c(S)_{w_{Chow} \ge m}$) whenever for any $j \in \mathbb{Z}$ we have $(W_{\le m+j} H_j^{t_l})(M) = H_j^{t_l}(M)$ (resp. $(W_{\le m+j-1} H_j^{t_l})(M) = 0$).

4. $M \in DM_c(S)_{w \le m}$ (resp. $M \in DM_c(S)_{w \ge m}$) whenever for any $n \in \mathbb{Z}$ we have $(W_{m+n} H^{et}_{\mathbb{Q}_l,n})(M) = H^{et}_{\mathbb{Q}_l,n}(M)$ (resp. $(W_{m+n-1} H^{et}_{\mathbb{Q}_l,n})(M) = 0$).

5. If $M \in ObjChow(S) (\subset ObjDM_c(S))$, then it can be decomposed into a direct sum of objects of $MM_j(S)[-j]$; this decomposition is unique up to a (non-unique) isomorphism.

6. If $M \in ObjMM_m(S)$, then it can be decomposed as a direct sum of simple objects of $MM_m(S)$; this decomposition is unique up to an isomorphism.

II Let S be of finite type over $\operatorname{Spec} \mathbb{F}_p$ *(for $p \neq l$); $M \in DM_c(S)^{t_l=0}$. Consider the weights for $D^b_c Sh^{et}(S, \mathbb{Q}_l)$ defined in §5 of [BBD82] (cf. Proposition 2.3.1(I)). Then $M \in DM_c(S)_{w_{Chow} \le m}$ (resp. $M \in DM_c(S)_{w_{Chow} \ge m}$) whenever $\mathcal{H}^{et}_{\mathbb{Q}_l}(M)$ is of weight $\le m$ (resp. of weight $\ge m$).*

Proof. I(1–3,5): Immediate from Proposition 1.4.2(II).

4. First we note that $H^{et}_{\mathbb{Q}_l,n}(M) \cong H^{et}_{\mathbb{Q}_l,0}(H_n^{t_l}(M))$. Applying assertion I3 we obtain: we may assume that $M \in DM_c(S)^{t_l=0}$ and consider only $m = 0$. Then applying Proposition 1.3.5(II2) for $F = H^{et}_{\mathbb{Q}_l,0}$ we obtain the result (we also use Proposition 1.3.5(II2) in order to relate the weight filtration for MM with Chow-weight spectral sequences).

6. Immediate from the semi-simplicity of MM_m.

II Immediate from assertion I(5) along with Theorem 2.3.3(I).

□

Remark 3.3.2. 1. $MM_m(S)$ could be called the category of *pure motives of weight m* (over S).

2. Consider a category MS of 'homological S-motives' whose objects are $DM_c(S)_{w_{Chow}=0}$, and

$$MS(M,N) = \operatorname{Im}(DM_c(S)(M,N) \to \bigoplus_{m\in\mathbb{Z}} Sh^{et}_{per}(S,\mathbb{Q}_l)(H^{et}_{\mathbb{Q}_l,m}(M), H^{et}_{\mathbb{Q}_l,m}(N)).$$

We conjecture that it is (anti)-isomorphic to the category $M(S)$ described in Definition 5.9 of ibid. (cf. Remark 2.1.2 of [Bon10c]).

We obtain: if a nice t_l exists over S, then MS is isomorphic to the direct sum of $MM_m(S)$ (as additive categories). Hence we obtain that MS is semi-simple (cf. Theorem 5.13 of [CoH00]); so it could also be called the category of 'numerical motives'. It is also easily seen that for any $M \in DM_c(S)_{w_{Chow}=0}$ the kernel of the projection $MM(Z,Z) \to MS(Z,Z)$ is a nilpotent ideal (cf. Theorem 6.9 of ibid.).

3. So, we proved that $\mathcal{H}^{et}_{\mathbb{Q}_l}$ 'strictly respects weights' if S is of finite type over $\operatorname{Spec}\mathbb{F}_p$; this is also true for $\tilde{\mathcal{H}}^{et}_{\mathbb{Q}_l}$ if S is of finite type over $\operatorname{Spec}\mathbb{Q}$. In [Wil08] a similar statement was established unconditionally for *Artin-Tate* motives over number fields.

4. Assume that $K \in Obj D^b_c Sh^{et}(S,\mathbb{Q}_l)$ is semi-simple (i.e. that it is a direct sum of shifts of semi-simple objects of $Sh^{et}_{per}(S,\mathbb{Q}_l)$) and that K is a retract of $K' = \mathcal{H}^{et}_{\mathbb{Q}_l}(N)$ for some $N \in Obj DM_c(S)$ (under our assumptions, this is easily seen to be equivalent to K being *semi-simple of geometric origin* in the sense of §6.2.4 of [BBD82]). Then our assumptions yield: K is a retract of $\bigoplus_{n\in\mathbb{Z}} H^{et}_{\mathbb{Q}_l,0}(M_n)[n]$ for some $M_n \in Obj \bigoplus_{m\in\mathbb{Z}} MM_m(S)$.

Indeed, it suffices to verify this statement for a simple $K \in Obj Sh^{et}_{per}(S,\mathbb{Q}_l)$; hence we can assume that $N \in Obj MM(S)$. Then we have morphisms $K \to H^{et}_{\mathbb{Q}_l,0}(W_{\le m}N)$ for all $m \in \mathbb{Z}$. Since K is simple, these morphisms are either zero or embeddings; hence K is a retract of one of $H^{et}_{\mathbb{Q}_l,0}(Gr^W_m(N))$.

5. Using the results of (§1.2 of) [Bon12] (some of which were stated above) one can derive some more consequences from the existence of a nice t_l.

3.4 Reducing the existence of a nice t_l to the universal domain case

Now we are ready to prove our second main result.

Theorem 3.4.1. *Suppose that for any point of a (reasonable) equicharacteristic scheme S the category $DM_c(K)$ possesses a nice t_l. Then the same is also true for $DM_c(S)$.*

Proof. By Theorem 3.1.5, t_l for $DM_c(S)$ exists. It remains to verify that t_l is transversal to w_{Chow}. For any (fixed) $M \in DM_c(S)^{t_l=0}$ and $m \in \mathbb{Z}$ we should verify the existence of a nice decomposition of M (see Definition 1.4.1).

Again, we apply the Noetherian induction, and assume that the statement is fulfilled over any closed subscheme of S.

Let K be a generic point of S. Since $j_K^*[-d]$ is t-exact and j_K^* is weight-exact, a nice choice of (2) (with the corresponding $m_K = m - d$) exists for $j_K^* M[-d]$ (in $DM_c(K)$; see Theorem 2.2.1(III)). By loc. cit. and Lemma 3.2.1, there exist an open embedding $j : U \to S$ (U contains K) along with a nice choice

$$A \stackrel{f'}{\to} j^*M \stackrel{g'}{\to} B \tag{6}$$

of (2).

We verify that this choice can be lifted to a one for M. We apply $j_{!*}$ to (6). Since $j_{!*}$ preserves monomorphisms and epimorphisms (see Proposition 1.5.3(IV5), we obtain a three-term complex as in (4) (i.e. $f = j_{!*}f'$ is monomorphic, and $g = j_{!*}g'$ is epimorphic). The middle-term homology object H_{mid} of the complex obtained belongs to $i_*DM_c(Z)$ by part IV3 of loc. cit. Since i_* is t- and weight-exact, the inductive assumption yields that a nice choice of (2) exists for H_{mid}. Now suppose that $j_{!*}(w_{\ge m+1}j^*M) \in DM_c(S)_{w\ge m+1}$ and $j_{!*}(w_{\le m}j^*M) \in DM_c(S)_{w\le m}$. Then we can choose 'trivial' nice decompositions for these objects; hence Proposition 1.4.2(I) would yield that a nice decomposition exists for $j_{!*}j^*M$. Now, applying Proposition 1.4.2(I) again along with Proposition 1.5.3(IV4) we obtain that a nice decomposition exists for M also.

Hence it remains to verify that $j_{!*}$ maps $DM_c(U)^{t_l=0} \cap DM_c(U)_{w\ge m+1}$ into $DM_c(S)_{w\ge m+1}$, and maps $DM_c(U)^{t_l=0} \cap DM_c(U)_{w\le m}$ into $DM_c(S)_{w\le m}$.

We fix some $M \in DM_c(U)^{t_l=0} \cap DM_c(U)_{w\ge m+1}$ (resp. $M \in DM_c(U)^{t_l=0} \cap DM_c(U)_{w\le m}$). Since $j^*j_{!*}M \cong M$, it suffices to verify that $i^!j_{!*}M \in DM_c(Z)_{w_{Chow}\ge m+1}$ (resp. $i^*j_{!*}M \in DM_c(Z)_{w_{Chow}\le m}$).

The inductive assumption for Z reduces the latter fact to a certain calculation of weight filtrations for $H^{et}_{\mathbb{Q}_l,n}$ of the corresponding motives; see Proposition 3.3.1(I4). In this form the statement follows immediately from Proposition 1.5.3(2) and Theorem 2.3.3(II2) (along with Proposition 2.1.1(14).

□

Remark 3.4.2. 1. Our arguments demonstrate that the notions of weight structure and of its transversality with t-structures are really important for the study of the 'weight filtration' of $DM_c(S)^{t_l=0}$ (cf. §4.1 below). Indeed, it seems that one cannot apply our gluing argument in the setting of filtered abelian categories (though possibly one could find a way to apply some of the corresponding arguments of [CoH00] in our context).

2. In contrast to the setting of the Theorem 3.1.5, we cannot prove the niceness of t when S is not a scheme over a field (without assuming Conjecture 2.3.2). The problem is that the weight-exactness of $j_{!*}$ does not follow

from the (Noetherian) inductive assumption considered in the proof of the theorem. Indeed, let $S = \operatorname{Spec}\mathbb{Z}_{(p)}$ (for a prime $p \neq l$); then one can glue $t_l(\operatorname{Spec}\mathbb{F}_p)$ with any 'shift' of $t_l(\operatorname{Spec}\mathbb{Q})$ (here we assume that $t_l(\operatorname{Spec}\mathbb{F}_p)$ and $t_l(\operatorname{Spec}\mathbb{Q})$ exist, and consider $(DM_c(\operatorname{Spec}\mathbb{Q})^{t'\leq 0}, DM_c(\operatorname{Spec}\mathbb{Q})^{t'\geq 0}) = (DM_c(\operatorname{Spec}\mathbb{Q})^{t_l\leq i}, DM_c(\operatorname{Spec}\mathbb{Q})^{t_l\geq i})$ for any $i \in \mathbb{Z}\setminus\{0\}$). Then the niceness of t_l over $\operatorname{Spec}\mathbb{Q}$ is equivalent to the niceness of t'; yet it seems highly improbable for $j_{!*}$ to be weight-exact for the weight structure obtained via this 'shifted gluing'. Hence in order to control the niceness of t_l for S in this case, one needs some 'extra' information on it. It seems quite reasonable to control motives via their homology; to this end we needto extend Theorem 2.3.3(II) to this case (see §7 of [Wil12] for some alternative arguments).

Now we want to prove that it suffices to verify the niceness of t_l over universal domains (only). To this end we need the following lemma.

Lemma 3.4.3. *Let $b : X \to \operatorname{Spec} K$ be a pro-smooth affine (separated) morphism (K is a field). Then for any $m \in \mathbb{Z}$, $M \in Obj DM_c(K)$, we have: $b^*M \in DM_c(X)_{w_{Chow}\leq m}$ (resp. $b^*M \in DM_c(X)_{w_{Chow}\geq m}$) whenever $M \in DM_c(K)_{w_{Chow}\leq m}$ (resp. $M \in DM_c(K)_{w_{Chow}\geq m}$).*

Proof. Theorem 2.2.1(II3) yields one of the implications.

Conversely, let $b^*M \in DM_c(X)_{w_{Chow}\leq m}$ (resp. $b^*M \in DM_c(X)_{w_{Chow}\geq m}$). Loc.cit. along with the definition of w_{Chow} easily yields: there exists a finite type smooth morphism $s : S \to \operatorname{Spec} K$ such that $s^*M \in DM_c(S)_{w_{Chow}\leq m}$ (resp. $s^*M \in DM_c(S)_{w_{Chow}\geq m}$). Then by Remark 2.2.2(2) of [Bon10c] we also have $s^!M \in DM_c(X)_{w_{Chow}\leq m}$ (resp. $s^!M \in DM_c(X)_{w_{Chow}\geq m}$). Choose a finite extension K'/K and a morphism $i : \operatorname{Spec} K' \to S$. Then $(i\circ s)^* = (i\circ s)^!M$; hence $(i\circ s)^*M \in DM_c(\operatorname{Spec} K')_{w_{Chow}\leq m}$ (resp. $(i\circ s)^*M \in DM_c(\operatorname{Spec} K')_{w_{Chow}\geq m}$); here we apply Theorem 2.2.1(II1). If b' is the structure morphism $\operatorname{Spec} K' \to \operatorname{Spec} K$, then we have $M = (b'\circ i\circ s)^*M$, whereas loc. cit. yields that $(b'\circ i\circ s)^*M \in DM_c(K)_{w_{Chow}\leq m}$ (resp. $(b'\circ i\circ s)^*M \in DM_c(K)_{w_{Chow}\geq m}$).

□

Corollary 3.4.4. *1. Suppose that a nice t_l exists over a universal domain K of characteristic $p > 0$. Then a nice t_l exists over any (reasonable) $S/\operatorname{Spec}\mathbb{F}_p$.*

2. Suppose that t_l exists over the field of complex numbers. Then a nice t_l exists over any (reasonable) $\operatorname{Spec}\mathbb{Q}$-scheme.

Proof. 1. The proof is rather similar to that of Corollary 3.2.4 (along with Proposition 3.2.3). We will only sketch it outlining the difference.

Again, it suffices to verify: if the transversality property is fulfilled for motives over a field L, then it is fulfilled over any its algebraically closed subfield, and over its subfield K such that the extension L/K is algebraic.

Both of these statements can be proved using the arguments in the proof of loc. cit. Indeed, by Proposition 1.4.2(III), we should verify that for any $M \in ObjDM_c(K)$ we have $(W_m H_0^{t_l})(M) \in DM_c(K)_{w_{Chow} \le m}$ and $M/(W_m H_0^{t_l})(M) \in DM_c(K)_{w_{Chow} \ge m+1}$ (for all $m \in \mathbb{Z}$). This can be easily done by combining the arguments from the proof of Corollary 3.2.4(1) with Lemma 3.4.3; note that $f_* = f_!$ is weight-exact if f is a finite morphism.

2. The statement is immediate from the previous assertion along with Proposition 1.5 of [Bei10]. $\square$

Remark 3.4.5. 1. It is also easily seen that if t_l is nice over S, it is also nice over all of its subschemes and residue fields. Indeed, it suffices to note that for any open immersion i and (the complimentary) closed embedding j the functors i_* and j^* are exact with respect to t_l and w_{Chow}, whereas i_* is a full embedding, j^* is a localization functor, and $\operatorname{Im} i_* = \operatorname{Ker} j^*$.

Certainly, this observation is far from being very exciting; yet it will make some of the formulations in §4.2 nicer.

2. Remark 2.1.4 of [Bon12] describes a funny way to produce new examples of transversal weight and t-structures (out of 'old' ones for a triangulated $\underline{C}$). To this end one should consider the so-called 'truncated categories' $\underline{C}_N$ (that are 'usually' defined for all $N \ge 0$). For our t, w, $\underline{C} = DM_c(S)$, we have $\underline{C}_0 = K^b(Chow(S))$. So (if certain 'standard' conjectures as listed in §4.1 below hold) this category shares several nice properties with $DM_c(S)$; this statement does not seem to be obvious.

4 Supplements

In §4.1 we verify that the existence of t_l and its niceness (over an equicharacteristic scheme S) follow from certain (more or less) 'standard' motivic conjectures (over algebraically closed fields; here we use certain lists of those taken from §1 of [Bei98] and §2 of [Han99]).

In §4.2 we note that our results yield a certain 'motivic Decomposition Theorem' (modulo the conjectures mentioned). In particular, we characterize pure motives over S in terms of those over its residue fields. This enables us to calculate $K_0(DM_c(S))$.

In §4.3 we extend (somehow) our results from the case of $\operatorname{Spec} \mathbb{Z}[\frac{1}{l}]$-schemes to the case of $\operatorname{Spec} \mathbb{Z}$-ones, and prove that the t-structure obtained does not depend on the choice of the corresponding l's. Here we need to

assume that the numerical equivalence of cycles is equivalent to $\mathbb{Q}_{l'}$-adic homological one (for any $l' \in \mathbb{P}$ and over universal domains of characteristic $\neq l', 0$).

4.1 Relating the existence of a (nice) t_l with 'standard' motivic conjectures

First we address the question: which (more or less) 'classical' motivic conjectures ensure the existence of t_l over S, that is nice if S is an equicharacteristic scheme. By the virtue of the results above, we only have to consider motives over universal domains. So we consider motives over some universal domain K of characteristic $p \neq l$ (p is either a prime or 0); recall that $DM_c(K) \cong DM_{gm}(K)$. None of the results of this paragraph are essentially original (unless we combine them with some of other our results).

Proposition 4.1.1. *The existence of a nice t_l for $DM_{gm}(K)$ is equivalent to (the conjunction of widely believed to be true) conjectures A–C of §1.2 of [Bei98].*

Proof. Conjectures A and B of loc. cit. state that $\mathbb{Q}_l$-adic étale cohomology on $DM_c(K)$ is strictly compatible with a certain t-structure (which we will denote by t_{MM}) for it. Now, it is easily seen that $t_{MM} = t_l$. Indeed, composing the étale cohomology with Poincare duality for $DM_c(K) = DM_{gm}(K)$ one obtains (a certain version of) étale homology for it. Note here that the Poincare duality for $DM_{gm}(K)$ exists for K of any characteristics (by an argument of M. Levine described in Appendix B of [HuK06]).

Lastly, Conjecture C of [Bei98] states that the homology objects for motives of smooth projective varieties (over K) with respect to t_l are semi-simple in $\underline{Ht}_l$. Then the same assertion is true for arbitrary Chow motives. Now, Proposition 1.4(ii) of [Bei10] yields the existence of the corresponding Chow-Kunneth decompositions (see Remark 1.4.3); hence $H_i^{t_l}[-i]$ of a motif of smooth projective P/K is a Chow motif (also). Hence t_l is nice (over K) by Proposition 1.4.2(V).

The converse implication is even easier (and is not really interesting for us).

□

Remark 4.1.2. 1. Here and throughout this paper we use the following observation: though the author doesn't know whether all possible versions of the ($\mathbb{Q}_l$-) étale homology realization for motives over a field K are isomorphic, one can still be sure that all of them yield the same t_l. Indeed, we have spectral sequences $T(-, H^{et}_{\mathbb{Q}_l,0})$ (for any version of $H^{et}_{\mathbb{Q}_l,0}$) that degenerate at E_2

(in this case; see Theorem 2.3.3(II1)). Hence $H^{et}_{\mathbb{Q}_l,m}(M)$ vanishes (for some $m \in \mathbb{Z}$) whenever $E_2^{p,m-p}(T) = 0$ for any $p \in \mathbb{Z}$. Now, in order to calculate $E_2^{p,m-p}(T)$ it suffices to know the restriction of $\mathcal{H}^{et}_{\mathbb{Q}_l}$ to $Chow(K)$ (see Proposition 1.3.2(I)), and certainly the latter does not depend on the choice of the version for $\mathcal{H}^{et}_{\mathbb{Q}_l}$.

2. Combining Proposition 3.2.3(2) with this spectral sequence argument, we obtain that the conservativity of the étale realization of motives (over fields or over general $\mathbb{Z}[\frac{1}{l}]$-base schemes) follows from the following conjecture: if K is an algebraically closed field of characteristic distinct from l, then any morphism of $Chow(K)$-motives that yields an injection on their étale (co)homology, splits (cf. Proposition 7.4.2 of [Bon09]). Note that the latter conjecture easily follows from the niceness of t_l over K (since t_l 'splits' Chow motives into direct sums of objects of semi-simple categories $MM_m[-m]$, whereas $\mathcal{H}^{et}_{\mathbb{Q}_l}$ is conservative on $MM_m[-m]$). Now, by the virtue of the results below, the niceness of t_l follows from 'standard' conjectures. Hence, there is a good reason to believe this ('Chow-splitting') conjecture.

Respectively, it could be interesting to study the connection of our results with those of [Ayo07].

Now we verify (briefly) the following statement: the existence of a nice t_l over K follows from the conjectures stated in §2 of [Han99]. Being more precise, one needs the standard conjecture D of loc. cit. (that $\mathbb{Q}_l$-homological equivalence of cycles coincides with numerical one), and Murre's conjectures A,D, and Van.

First we note that Murre's conjecture A yields the existence of Chow-Kunneth decompositions of motives of smooth projective varieties (over K) i.e. any such motif can be decomposed (in $Chow(K)$) into a direct sum of motives each of those has only one non-zero $\mathbb{Q}_l$-adic (co)homology group. Here and below we can consider $\mathcal{H}^{et}_{\mathbb{Q}_l}$ instead of étale cohomology; cf. the proof of Proposition 4.1.1. Next, (the proof of) Proposition 2.4 of [Han99] implies that the conjectures mentioned imply all the remaining Murre's conjectures (we can apply loc. cit. here since the Lefschetz type standard conjecture B used in its proof follows from standard conjecture D by the main result of [Smi97]). We define $MM_m(K)$ as the subcategory of $Chow[m] \subset DM_{gm}(K)$ consisting of objects whose $\mathbb{Q}_l$-étale (co)homology is concentrated in degree 0.

Proposition 2.3 of [Han99] yields that the categories $MM_m(K)$ are isomorphic to the corresponding pieces of the category of $\mathbb{Q}_l$-étale homological motives. Conjecture D embeds them into the category of numerical motives (which is semi-simple by the main result of [Jan92]); hence they are semi-simple also. Next, the arguments used for the proof of Proposition 2.9

of [Han99] yield for $MM_m(K)$ the orthogonality conditions of Proposition 1.4.2(IV). Besides, by Lemma 1.1.1(6) of [Bon12] these conditions also yield that the category $C \subset Chow(K)$ with $ObjC = \bigoplus ObjMM_m(K)[-m]$ is idempotent complete; hence $C = Chow(K)$. Since $\langle Chow(K)\rangle = DM_{gm}(K)$ (see Proposition 2.1.1(2)), Proposition 1.4.2(IV) yields that $DM_{gm}(K)$ possesses a t-structure t_{MM} that is transversal to w_{Chow}. Since all objects of $\underline{Ht}_{MM}$ possess filtrations whose factors belong to $MM_m(K)$ (see Proposition 1.4.2(II4)), we obtain that $Obj\underline{Ht}_{MM} \subset DM_c(K)^{t_l=0}$; hence $\mathcal{H}^{et}_{\mathbb{Q}_l}$ is t-exact with respect to t_{MM}. Moreover, Murre's conjecture Van yields that $\mathcal{H}^{et}_{\mathbb{Q}_l}$ does not kill non-zero objects of $MM_m(K)$ (since it does not kill non-zero Chow motives). Then Proposition 1.4.2(II4) also implies that $Obj\underline{Ht}_{MM} = DM_c(K)^{t_l=0}$; cf. the proof of Proposition 3.3.1(I4). Hence $t_{MM} = t_l$ (see Remark 1.2.3(3)), and we obtain the result desired.

Remark 4.1.3. 1. Alternatively, one can prove the existence of the motivic t-structure for $DM_{gm}(K)$ using the arguments from the proof of Theorem 3.4 of [Han99], whereas (the proof of) Proposition 2.8 of ibid. allows us to verify the conditions of Proposition 1.4.2(V) that ensure (in this case) that $w_{Chow}(K)$ is transversal to t_l.

2. For a characteristic 0 field K one can apply the results of Corti and Hanamura directly (after replacing étale cohomology by $\mathcal{H}^{et}_{\mathbb{Q}_l}$ using Poincare duality). Indeed, it was proved in §4 of [Bon09] that in this case Hanamura's triangulated category of motives is isomorphic to $DM_{gm}{}^{op}$.

3. It seems that the existence of t_l without any additional assumptions does not imply its niceness (at least, easily) over positive characteristic fields (one also needs to assume the Hodge standard conjecture or the conjectures mentioned above for this matter). Over (Spec $\mathbb{Q}$-schemes and) characteristic 0 fields one does not need any extra assumptions; cf. Corollary 3.4.4(2) (and also Proposition 2.2 of [Bei10]).

4.2 Our 'motivic Decomposition Theorem'

Our results easily yield a motivic version of the celebrated Topological Decomposition Theorem (for perverse sheaves; see Remark 4.2.4). In particular, we characterize pure motives (see Remark 3.3.2(1)) 'pointwisely'. In order to formulate our results, we need a certain intermediate image functor for j_K, where $K \in \mathcal{S}$.

First let K be a generic point of S of dimension d (see the Notation). Suppose that a nice t_l exists over S; then it also exists over K (see Remark 3.4.5(1)). We define $j^d_{K!*}$ for $M \in MM_m(K)$ ($m \in \mathbb{Z}$) in the following way. First we lift $M[d]$ to a certain $M_U \in MM_{m+d}(U)$ for some open $U \subset S$,

$K \in U$; here we use Lemma 3.2.1 and Theorem 2.2.1(III) (cf. the proof of Theorem 3.4.1). M_U is semi-simple in $MM(U)$ (see Proposition 3.3.1(I6)); and we take M'_U being the sum of those components of M_U that are not killed by j_K^{U*} (for the corresponding morphism $j_K^U : K \to U$; note that M'_U is determined by M_U uniquely up to an isomorphism). Lastly, we set $j^d_{K!*}M = j_!M'_U$, where $j : U \to S$ is the corresponding open immersion.

Now, let K be an arbitrary point of S of dimension d, whose closure is $Z \subset S$; $i : Z \to S$ is the corresponding embedding. Then we lift $M[d]$ to $MM_{m+d}(Z)$ using the procedure described above, and then apply i_* in order to obtain $j^d_{K!*}M$. Certainly, here we use t_l- and w_{Chow}-exactness of $i_* = i_!$. Besides, note: if we denote the composite immersion $U \to Z \to S$ by j, then we would have

$$j^d_{K!*}M = \mathrm{Im}(H_0^{t_l} j_! M'_U \to H_0^{t_l} j_* M'_U) \tag{7}$$

in this case (also).

Lemma 4.2.1. *1. $j^d_{K!*}M$ does not depend on any choices (if a nice t_l exists over S). Moreover, $j^d_{K!*}$ yields a full embedding (of categories).*

2. $j^d_{K!*}M$ is functorially characterized by following condition: it is a semi-simple lift of M to $MM_{m+d}(S)$ none of whose direct summands are killed by j^*_K.

Proof. This is an easy consequence of Proposition 1.5.3(IV7-9). ☐

Remark 4.2.2. Alternatively, one could try to apply here the (somewhat parallel) arguments of §5 of [Sch12]. Yet certain adjustments are certainly needed to do this (along with certain results of §2.3 of [Bon10c]).

Proposition 4.2.3. *Let a nice t_l exist over S. Then for any $m \in \mathbb{Z}$ any object of $MM_m(S)$ can be decomposed as a direct sum of $j^{d_i}_{K_i!*}M_i$ for $K_i \in \mathcal{S}$ being of dimension d_i, and $M_i \in MM_{m-d_i}(K_i)$ being indecomposable objects. This decomposition is unique up to an isomorphism. Moreover, $M_i \cong H^{t_l}_{-d_i}(j^*_{K_i}M)$, whereas K_i can be characterized by the condition that $H^{t_l}_{-d_i}(j^*_{K_i}M) \neq 0$.*

Proof. First we verify that M can be decomposed into a direct sum of some $j^{d_i}_{K_i!*}M_i$ (somehow). Since $MM_m(S)$ is semi-simple, it suffices to prove: if M is indecomposable, then it can be presented as $j^{d_M}_{K_M!*}M_K$ for some $K_M \in \mathcal{S}$ of dimension d_M and $M_K \in MM_{m-d}(K_M)$. We prove this statement by noetherian induction (applying Remark 3.4.5(1) again).

We take K being a generic point of S. If $j^*_K M \neq 0$, Lemma 4.2.1 immediately implies that we can take $K_M = K$, $M_K = j^*_K M[-d]$. Conversely, if $j^*_K M = 0$, then there exists an open immersion $j : U \to S$ ($K \in U$) such that

$j^*M = 0$. Hence for the complimentary closed embedding $i : Z \to S$ there exists a (simple) $M_Z \in MM_m(Z)$ such that $M \cong i_*M_Z$ (since i_* is weight- and t-exact). Hence it suffices to apply the inductive assumption to M_Z (see (7)).

It remains to verify: for $K, K' \in \mathcal{S}$ of dimensions d and d' respectively, $M \in MM_n(K)$ ($n \in \mathbb{Z}$) we have: $H^{t_l}_{-d'}(j^*_{K'} j^d_{K!*}(M)) = 0$ if $K' \neq K$ and $= M$ otherwise. We consider three cases here: 1) $K' = K$, 2) K' belongs to the closure Z of K in S, and 3) K' does not belong to Z.

Denote the embedding of Z into S by i; denote the complimentary immersion by j and the morphism $K \to Z$ by j^Z_K. In case 1) it suffices to note that $j^{Z,d}_{K!*}M$ is a lift of $M[d]$ to $DM_c(Z)$, whereas $i^*i_* = 1_{DM_c(Z)}$. In case 3) it suffices to note that $j^*_{K'}$ factorizes through j and that $j^*i_* = 0$. In case 2) we can assume that $Z = S$ (since $i^*i_* = 1_{DM_c(Z)}$); then our claim easily follows from Proposition 1.5.3(IV2).

□

Remark 4.2.4. 1. The 'usual' Topological Decomposition theorem (see Theorem 5.7 of [CoH00]) states (for S being a variety over a field): if $X \to S$ is a proper morphism, X is regular, then $f_*\mathbb{Q}_{lX} \in Obj D^b_c Sh^{et}(S, \mathbb{Q}_l)$ splits as a direct sum of its t-homology, whereas its homology (perverse) sheaves can be presented as direct sums of intermediate images of pure $\mathbb{Q}_l$-local systems supported on some subvarieties of S. We verify that this decomposition can be lifted to $DM_c(S)$ (hence, we can improve Theorem 5.14 of [CoH00]) even if we replace $\mathbb{Q}_{lX}$ here by $K = \mathcal{H}^{et}_{\mathbb{Q}_l}(N)$ for any $N \in H^{et}_{\mathbb{Q}_l,0}(MM_n(X))$ (for some $n \in \mathbb{Z}$) and do not demand X to be regular.

First we note that $f_*K = \mathcal{H}^{et}_{\mathbb{Q}_l}(f_*N)$ (see Proposition 2.1.1(14)), whereas $f_*N \in DM_c(S)^{w_{Chow}=n}$ (see Theorem 2.2.1(II1)). By Proposition 3.3.1(I5) we obtain that t_l splits f_*N into a direct sum of objects of $MM_j(S)[n-j]$.

Hence in order to fulfill our goal it suffices to verify (by the virtue of Proposition 4.2.3; for an $m \in \mathbb{Z}$) for any $M \in Obj MM_m(S)$ that the (perverse) homology of the corresponding $j^{d_i}_{K_i!*}M_i$ can be presented as the the intermediate image of a $\mathbb{Q}_l$-local system supported on some regular connected subvariety U_i of S, whereas K_i is the generic point of U_i (cf. Theorem 4.3.1(ii) of [BBD82]). As we have verified above, for any such U_i the motif $j^{d_i}_{K_i!*}M_i$ can be presented as $j_{U_i!*}M_{U_i}$ for some $M_{U_i} \in Obj MM_m(U_i)$ (here we denote by $j_{U_i!*}$ the composite of the intermediate image functor for the embedding of U_i into its closure Z_i with the direct image $DM_c(Z_i) \to DM_c(S)$). We should prove that we can choose U_i, M_{U_i} such that $\mathcal{H}^{et}_{\mathbb{Q}_l}(M_{U_i}) \in Obj Sh^{et}_{per}(S, \mathbb{Q}_l)$ is a local $\mathbb{Q}_l$-system on U_i.

By Theorem 2.2.1(III2), we can assume (if we choose U_i to be small enough) that there exist: a regular scheme U'_i, a finite universal homeomor-

phism $g : U'_i \to U_i$, a smooth projective morphism $h : P_i \to U'_i$, and an $s \in \mathbb{Z}$ such that M_{U_i} is a retract of $(g \circ h)_* \mathbb{Q}_P(s)[2s+m]$. It remains to note that the homology sheaves of $(g \circ h)_* \mathbb{Q}_{lP_i}(s) \in Obj D^b_c Sh^{et}(U_i, \mathbb{Q}_l)$ are pure local systems (since this is true for the 'canonical' homology of this derived category, and U_i is regular, we obtain that the perverse homology equals the canonical one).

2. More generally, consider K being a retract of an object K' of $\mathcal{H}^{et}_{\mathbb{Q}_l}(\bigoplus_{m,j\in\mathbb{Z}} MM_m(X)[j])$. As noted in Remark 3.3.2(4), this condition is fulfilled (in particular) if K is a semi-simple complex of geometric origin (in the sense of [BBD82]). Then f_*K' belongs to $\mathcal{H}^{et}_{\mathbb{Q}_l}(\bigoplus_{m,j\in\mathbb{Z}} MM_m(S)[j])$. Since local systems over subschemes of S yield Krull-Schmidt subcategories, we obtain that f_*K can be presented as a direct sum of retracts of $\mathcal{H}^{et}_{\mathbb{Q}_l}(j_{U_i!*}M_{U_i}[s_i])$ for (U_i, M_{U_i}, s_i) corresponding to f_*K'.

Thus we obtain a certain motivic analogue of Theorem 6.2.5 of [BBD82]. Yet note that (in contrast with loc.cit.) f_* does not preserve semi-simplicity of perverse sheaves in general. Indeed, even if we take $S = \operatorname{Spec} K$ for a (general) field K and a smooth projective X/K, then the étale cohomology of $X_{K^{sep}}$ need not be semisimple as $\mathrm{Gal}(K)$-representations (for example, for $K = \mathbb{Q}_p$ we do not have semi-simplicity for H^1_{et} of an elliptic curve with split multiplicative reduction; see Exercise 5.13 of [Sil94]).

Now we are (also) able to calculate $K_0(DM_c(S))$.

Corollary 4.2.5. *Define $K_0(DM_c(S))$ as a group whose generators are $[C]$, $C \in Obj DM_c(S)$; if $D \to B \to C \to D[1]$ is a distinguished triangle in $DM_c(S)$ then we set $[B] = [C] + [D]$.*

Let a nice t_l exist over S. Then $K_0(DM_c(S))$ is a free abelian group with a basis indexed by isomorphism classes of indecomposable objects of $MM_m(K_i)$ for K_i running through all elements of $\mathcal{S}$, m running through all integers.

Proof. By Proposition 1.2.6 of [Bon12], $K_0(DM_c(S))$ is a free abelian group with a basis indexed by isomorphism classes of indecomposable objects of $MM_m(S)$ (for m running through all integers). Now the result follows from Proposition 4.2.3 easily. □

4.3 Changing l; the case of Spec $\mathbb{Z}$-schemes

First we study the question when $t_{l'}$ exists and coincides with t_l for prime $l \neq l'$ (we fix the primes, and define $t_{l'}$ similarly to t_l).

Proposition 4.3.1. *1. Suppose that $\mathcal{H}^{et}_{\mathbb{Q}_l}$ and $\mathcal{H}^{et}_{\mathbb{Q}_{l'}}$ exist and coincide over universal domains of all characteristics $\neq l, l'$. Then they also exist and coincide over any reasonable* $\operatorname{Spec}\mathbb{Z}[\frac{1}{l}, \frac{1}{l'}]$*-scheme S.*

In particular, this assertion holds if $\mathcal{H}^{et}_{\mathbb{Q}_l}$ and $\mathcal{H}^{et}_{\mathbb{Q}_{l'}}$ are nice over the universal domains mentioned.

2. Suppose that for any prime $p \neq l, l'$ there exists a universal domain K of characteristic p such that: there exists a nice t_l for $DM_c(K)$, and $\mathcal{H}^{et}_{\mathbb{Q}_{l'}}$-homological equivalence of cycles coincides with the numerical equivalence one over K.

Then t_l and $t_{l'}$ exist and coincide on $DM_c(S)$ for any reasonable $\operatorname{Spec} \mathbb{Z}[\frac{1}{l}, \frac{1}{l'}]$-scheme S.

Proof. 1. First we note that the niceness of t_l and t'_l for $DM_c(K)$ yields that they coincide (on $DM_c(K)$) by Proposition 4.5 of [Bei98].

So, it remains to verify that t_l and t'_l exist and coincide over S if this is true over universal domains (of all characteristics $\neq l, l'$).

By Corollary 3.2.4(1), t_l and $t_{l'}$ exist for $DM_c(S)$. By Lemma 3.1.3(7), it suffices to verify that $DM_c(F)^{t_{l'} \leq 0} = DM_c(F)^{t_l \leq 0}$ and $DM_c(F)^{t_{l'} \geq 0} = DM_c(F)^{t_l \geq 0}$ for any point F of S. An argument similar to the one used in the proof of Corollary 3.2.4(1) yields that we can replace F by one of our universal domains (of the same characteristic).

2. By the previous assertion, it suffices to verify that t_l coincides with $t_{l'}$ over K.

Since all $MM_i(K) = DM_c(K)^{t_l=0} \cap DM_c(K)_{w_{Chow}=i}$ are semi-simple, we obtain that $\mathcal{H}^{et}_{\mathbb{Q}_l}$-homological equivalence of cycles is equivalent to numerical equivalence (over K). Indeed, consider the functor $\bigoplus Gr^{t_l}_m : Chow(K) \to \bigoplus MM_m(K)$ that is given by the direct sum of all (shifted) t_l-homology of Chow motives; see Remark 3.3.2(2). It kills exactly those morphisms of Chow motives that are $\mathbb{Q}_l$-homologically equivalent to zero as cycles (since $\mathcal{H}^{et}_{\mathbb{Q}_l}$ does not kill non-zero morphisms in $MM_m(K)$). It remains to note that $\mathbb{Q}_l$-homological equivalence is finer than the numerical one, and the category of numerical motives (over K) is semi-simple (see [Jan92]). Besides, $\mathcal{H}^{et}_{\mathbb{Q}_{l'}}$ is also conservative on $MM_m(K)$ for any $m \in \mathbb{Z}$.

Now, for a smooth projective P/K Proposition 5.4 of [Kle94] implies (in our setting) that the (Chow)-Kunneth decomposition of the motif of P coming from t_l is also its Kunneth decomposition with respect to $\mathcal{H}^{et}_{\mathbb{Q}_{l'}}$. Indeed, by loc. cit. the numerical equivalence classes of the corresponding projectors do not depend on the choice of a Weil (co)homology theory; note that (by the main result of [Smi97]) we can replace the Hodge standard conjecture by the Standard Conjecture D (see §4.1) in the assumptions of [Kle94]. Hence $\mathcal{H}^{et}_{\mathbb{Q}_{l'}}$ sends $MM_m(K)$ into $D^b_c Sh^{et}(K, \mathbb{Q}_{l'})^{t=0}$. Then Proposition 3.3.1(I2) easily yields that $DM_c(K)^{t_{l'}=0} = DM_c(K)^{t_l=0}$; boundedness yields that $t_{l'}$ coincides with t_l (see Remark 1.2.3(3)).

$\square$

Now suppose that S is not (necessarily) a $\operatorname{Spec}\mathbb{Z}[\frac{1}{l}]$-scheme. We note: $\mathcal{H}^{et}_{\mathbb{Q}_l}$ kills all motives 'supported on' $S \times_{\operatorname{Spec} Z} \operatorname{Spec}\mathbb{F}_p$. In order to overcome this difficulty we introduce the following definition.

Definition 4.3.2. Let $l \neq l'$ be primes. Consider the class $DM_c(S)^{t_{l,l'}\leq 0}$ (resp. $DM_c(S)^{t_{l,l'}\geq 0}$) consisting of those $M \in Obj DM_c(S)$ that satisfy: $\mathcal{H}^{et}_{\mathbb{Q}_l}(M) \in D^b_c Sh^{et}(S[1/l], \mathbb{Q}_l)^{t_l \leq 0}$ and $\mathcal{H}^{et}_{\mathbb{Q}_{l'}}(M) \in D^b_c Sh^{et}(S[1/l'], \mathbb{Q}_{l'})^{t_{l'} \leq 0}$ (resp. $\mathcal{H}^{et}_{\mathbb{Q}_l}(M) \in D^b_c Sh^{et}(S[1/l], \mathbb{Q}_l)^{t_l \geq 0}$ and $\mathcal{H}^{et}_{\mathbb{Q}_{l'}}(M) \in D^b_c Sh^{et}(S[1/l'], \mathbb{Q}_{l'})^{t_{l'} \geq 0}$).

If $(DM_c(S)^{t_{l,l'}\leq 0},\ DM_c(S)^{t_{l,l'}\geq 0})$ yield a t-structure for $DM_c(S)$, we will say that (the t-structure) $t_{l,l'}$ exists over S.

Now we observe that 'standard' conjectures imply: $t_{l,l'}$ exists over any (reasonable) S and does not depend on l. We try to formulate this result concisely; some more (somewhat stronger) statements of this sort can also be easily proved.

Proposition 4.3.3. *Let l, l' be as above.*

1. Suppose that a nice t_l exists over any universal domain of any characteristic $\neq l$; a nice $t_{l'}$ exists over a universal domain of characteristic l, and that $\mathcal{H}^{et}_{\mathbb{Q}_{l'}}$-homological equivalence of cycles is equivalent to numerical equivalence over any universal domain of any characteristic $\neq l', 0$. Then $t_{l,l'}$ is a t-structure over any (reasonable) S.

2. Suppose moreover that for any prime p (distinct from l, l') we have: $\mathcal{H}^{et}_{\mathbb{Q}_p}$-homological equivalence of cycles coincides with the numerical equivalence one over any universal domain of any characteristic $\neq p$. Then $t_{l,l'}$ does not depend on the choice of the pair l, l'.

Proof. 1. It is easily seen that $t_{l,l'}$ can be characterized similarly to Lemma 3.1.3(7) i.e.: $M \in DM_c(S)^{t_{l,l'}\leq 0}$ (resp. $M \in DM_c(S)^{t_{l,l'}\geq 0}$) whenever for any $K \in \mathcal{S}$, K is of dimension d, we have $j_K^* M[-d] \in DM_c(K)^{t_{l,l'}\leq 0}$ (resp. $j_K^! M[-d] \in DM_c(K)^{t_{l,l'}\geq 0}$). Indeed, this is an easy consequence of loc. cit.

Next we note that over characteristic 0 universal domains the homological equivalence of cycles relation does not depend on the choice of l since it can be described in terms of singular (co)homology. Hence Proposition 4.3.1 yields that $t_l = t'_l$ over any field of characteristic $\neq l, l'$.

It easily follows that $t_{l,l'}(S)$ can be glued from t_l for $S[1/l]$ and $t_{l'}$ for $S \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_l$; see Proposition 1.5.3(II1) and Proposition 2.1.1(8).

2. It suffices to apply Proposition 4.3.1 (again) and the arguments described above.

$\square$

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